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MODELING GEOMETRIC NON-LINEARITIES IN THE FREE VIBRATION OF A PLANAR BEAM FLEXURE WITH A TIP MASS

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ABSTRACT

The objective of this work is to create an analytical framework to study the non-linear dynamics of beam flexures with a tip mass undergoing large deflections. Hamilton's principal is utilized to derive the equations governing the non-linear vibrations of the cantilever beam and the associated boundary conditions. Then, using a single mode approximation, these non-linear partial differential equations are reduced to two coupled non-linear ordinary differential equations. These equations are solved analytically using combination of the method of multiple time scales and homotopy perturbation analysis. Closed-form, parametric analytical expressions are presented for the time domain response of the beam around and far from its internal resonance state. These analytical results are compared with numerical ones to validate the accuracy of the proposed closed-form model. We expect that the qualitative and quantitative knowledge resulting from this effort will ultimately allow the analysis, optimization, and synthesis of flexure mechanisms for improved dynamic performance.

1. INTRODUCTION

Beams are important building blocks in many engineering structures ranging from micro/nano devices to macro-scale airplane wings, flexible satellites, and long span bridges [1]. Flexible beams are also one of the most important constitutive elements in flexure mechanisms. Flexure mechanisms provide guided motion via elastic deformation, instead of employing sliding or rolling joints, and are used in a variety of applications that demand high precision, minimal assembly, long operating life, or design simplicity [2]. Since they exhibit motion guidance as well as elastic behavior, flexure mechanisms are also ideally suited for single- and multi- axis resonators, energy harvesting devices, and high-speed scanners. However, large motion range in flexure mechanisms implies large elastic deflections of the constituent beams, which in turn give rise to geometric non-linearities. Even though commonly ignored, these non-linearities critically influence the dynamic characteristics of a flexure mechanism [3]. Depending on the application, the relevant dynamic characteristics could include vibrational mode shapes, flow of energy between modes, bandwidth or speed of response, dynamic range, command tracking, noise and disturbance sensitivity, closed-loop stability and robustness, etc. As a result, investigating the non-linear dynamical behavior of flexure mechanisms is of primary importance in their design.

In a flexure mechanism, the elastic motion provided via flexure beams is transferred to one or more moving stages, which can initially be modeled as concentrated masses. In fact, many flexure mechanisms can be represented as a system of interconnected beams with point masses. Therefore, a logical first step in investigating the non-linear dynamics of flexure mechanisms is to consider and understand the vibrational behavior of a simple Euler beam, with a tip mass at its end. Such a study is the focus of this paper.

In general, the sources of non-linearities may be geometric or material. The geometric non-linearity arises from arc-length conservation of the beam and large deformation curvatures at which the linear relationship between displacement field and strains no longer holds. Material non-linearity occurs when the stresses are non-linear functions of strains [4].

Because of its long, slender geometry, a typical beam flexure may be modeled using the Euler-Bernoulli beam theory. This theory assumes that plane cross-sections continue to remain plane and normal to the neutral axis after deformation [5], and has been successfully utilized to study the static, dynamic, and vibrational behavior of beams. In particular, large amplitude vibrations of beams have been extensively investigated both theoretically and experimentally in the literature. Crespo daSilva [6] formulated the non-linear differential equations of motion for Euler-Bernoulli beams experiencing flexure along two principal directions, along with torsion and extension. He presented a reduced-order analytical model for the non-linear dynamics of a class of flexible multi beam structures [7]. Nayfeh modeled the non-linear transverse vibration of beams with properties that vary along the length

[8]. Zaretzky et al. [9] experimentally investigated the non-linear modal coupling in the response of cantilever beams.

The presence of a tip mass on the beam changes the differential equations governing its deflection. This is because the inertial force exerted on the beam due to the presence of a concentrated mass is a function of the deflection itself. Large amplitude vibrations of beams with tip mass have also been investigated in the literature. Hijmissen and Horssen analyzed the weakly damped transverse vibrations of a vertical beam with a tip mass [10]. Zavodney and Nayfeh studied the non-linear response of a slender beam carrying a lumped mass to a principal parametric excitation [11]. But the axial dynamics of the beam, which can become important at large deflections, was not considered in these formulations.

This paper presents an analytical investigation of the non-linear in-plane oscillations of a flexure beam with a tip mass, while including axial stretching. The Homotopy Perturbation Method (HPM) is employed because it does not depend upon the assumption of small parameters in the non-linear equations and takes full advantage of the traditional perturbation methods as well as homotopy techniques. HPM has been used to investigate non-linear vibrations of beams in the recent literature. For example Moeenfard et al. used a combination of HPM and the modified Lindstedt-Poincare technique to analyze non-linear free vibrations of Timoshenko micro-beams [12]. In this paper, HPM is utilized in conjunction with the method of multiple time scale perturbation method to solve the non-linear dynamics of the beam.

2. PROBLEM FORMULATION

The beam with end-mass considered in this analysis is shown in Figure 1. The dashed line represents the un-deformed state, while the solid line represents a general deformed state. The gravitational field, if any, is assumed normal to the plane and is therefore does not affect the planar analysis considered here.

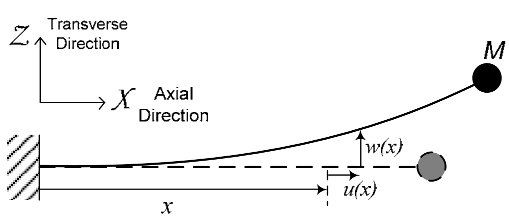

Figure 1 *Schematic view of a beam with a tip mass*

As the first step, the equations of motion and boundary conditions corresponding to the transverse and axial vibrations of a slender beam will be derived using the generalized Hamilton's principle. In the Euler–Bernoulli beam theory, plane cross-sections remain plane and perpendicular to the neutral axis after deformation, which implies that distortions due to shear are neglected. These assumptions are applicable for long and slender beams, with length much greater than the thickness [5]. Since the beam undergoes large deflections, the non-linear strain expression is used for calculating its strain energy. For a differential element on the centerline of the beam, the non-linear strain ε_{xx} is as follows [13]:

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{1}$$

where u and w are the displacements along X and Z axes, respectively.

Using equation (1), the axial strain of an element at distance z from the neutral axis, along the Z direction, may be expressed as follows [13].

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-z\left(\frac{\partial^2 w}{\partial x^2}\right) \tag{2}$$

Using equation (2), the strain energy of the beam for a linear elastic material would be

$$\begin{aligned}\pi&=\frac{E}{2}\iiint_V \varepsilon_{xx}^2\,dA.dx=\frac{E}{2}\iint\int_0^l\Bigg[\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)^2\\&+\left(\frac{\partial^2 w}{\partial x^2}\right)^2 z^2-2z\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\Bigg]dA.dx\\&=\frac{EA}{2}\int_0^l\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)^2dx+\frac{EI}{2}\int_0^l\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx\end{aligned} \tag{3}$$

where l is the un-deformed length of the beam, A is the area of the cross section, and I is the second moment of the area of the cross section about the neutral axis.

In long slender beams where $u(x,t)=O(w(x,t)^2)$, the axial inertia of the beam can be ignored compared with the concentrated inertial loads applied at the tip of the beam [4]. Assuming that axial damping is also negligible, the axial strain $\varepsilon_{xx}=\partial u/\partial x+(\partial w/\partial x)^2/2$ would remain constant along the length of the beam. In such a condition, the potential energy given in equation (3) can be simplified as

$$\begin{aligned}\pi&=\frac{EA}{2l}\left(\int_0^l\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)dx\right)^2\\&+\frac{EI}{2}\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx=\frac{EA}{2l}\Bigg(u(l,t)+\\&\frac{1}{2}\int_0^l\left(\frac{\partial w}{\partial x}\right)^2dx\Bigg)^2+\frac{EI}{2}\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx\end{aligned} \tag{3}$$

It can be shown that for an infinitesimal element of the beam, the ratio of the rotational kinetic energy to the translational kinetic energy is approximately of the order of $(h/l)^2$, where h is the thickness of the beam. Since for a long and slender beam $h \ll l$, the rotational kinetic energy may be ignored [4, 5]. Additionally, since in a planar beam flexure $u(x,t)$ is approximately two orders of magnitude smaller than $w(x,t)$, i.e. $u(x,t)=O(w(x,t)^2)$, the axial kinetic energy of a beam element is at least four orders of magnitude smaller than its transverse kinematic energy, and therefore may also be ignored [4]. Thus, the total kinetic energy is simply given by:

$$T = \frac{1}{2}\int_0^l \left(\frac{\partial w}{\partial t}\right)^2 \rho.A.dx + \frac{1}{2}M\left(\left(\frac{\partial u(l,t)}{\partial t}\right)^2 + \left(\frac{\partial w(l,t)}{\partial t}\right)^2\right) \quad (4)$$

where ρ is the material density and M is the tip mass.

Assuming that the beam is vibrating in viscously damped media and assuming that the axial damping is negligible with respect to transverse damping, the virtual external work done on the beam by distributed damping loads would be

$$\delta W_e = -\int_0^l c_t \frac{\partial w(x,t)}{\partial x} \delta w(x,t)\, dx \quad (5)$$

where c_t is the damping coefficient per unit length in the transverse direction.

Now using the generalized Hamilton's principle, the equations governing the non-linear dynamics of a beam undergoing large in-plane motions and the related geometric boundary conditions are obtained as follows.

$$\frac{\partial^2}{\partial x^2}\left(E.I\frac{\partial^2 w}{\partial x^2}\right) - \frac{E.A}{l}\left(u(l,t) + \frac{1}{2}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial t}\left(\rho.A.\frac{\partial w}{\partial t}\right) = -\frac{\partial}{\partial t}\left(M\frac{\partial w}{\partial t}\right)\hat{f}(l) \quad (6)$$

$$M\frac{d^2 u(l,t)}{dt^2} + \frac{E.A}{l}\left(u(l,t) + \frac{1}{2}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx\right) = 0 \quad (7)$$

$$w(0,t) = \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0 \quad (8)$$

In equation (6), $\hat{f}(x)$ is the Dirac delta function which is used to model the concentrated inertial load at $x = l$. Figure 2 shows a schematic view of the Dirac delta function.

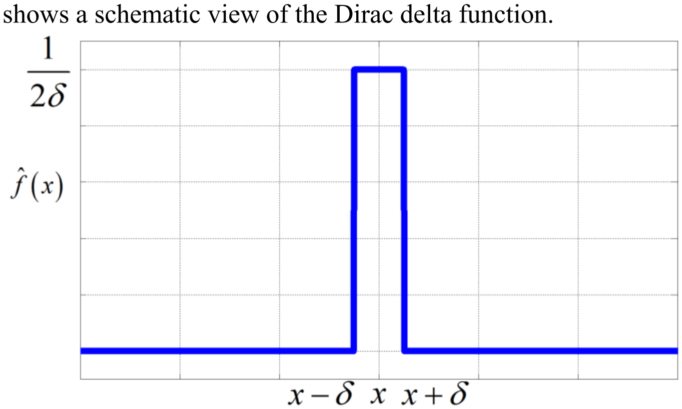

Figure 2 *Dirac delta function* $\hat{f}(x)$ *where* $\delta \to 0^+$

For convenience, the following dimensionless variables are introduced.

$$\hat{x} = \frac{x}{l} \quad (9)$$

$$\hat{w} = \frac{w}{l} \quad (10)$$

$$\hat{u} = \frac{u}{l} \quad (11)$$

$$\hat{t} = t\sqrt{\frac{E.I}{\rho.A.l^4}} \quad (12)$$

By substituting these dimensionless quantities into equations (6) and (7), dropping the hats, and assuming that $E.I$ is constant with respect to coordinate x, and $\rho.A$ and M are constant with respect to time, the following equations may be derived:

$$\frac{\partial^4 w}{\partial x^4} - \sigma_1\left(u(t) + \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} + C_t\frac{\partial w}{\partial t} + \sigma_2\frac{\partial^2 w(x,t)}{\partial t^2}\tilde{f}(1) = 0 \quad (13)$$

$$\lambda_1\frac{d^2 u(t)}{dt^2} + \left(u(t) + \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right) = 0 \quad (14)$$

where $u(t)$ is the normalized axial displacement of the tip mass and

$$\sigma_1 = \frac{Al^2}{I} \quad (15)$$

$$\sigma_2 = \frac{M}{\rho.A.l} \quad (16)$$

$$\lambda_1 = \frac{M}{\rho.A.l}\frac{I}{Al^2} \quad (17)$$

$$C_t = \frac{c_t l^2}{\sqrt{\rho.A.E.I}} \quad (18)$$

The first mode of a typical system is generally the most important one. When the system is excited, most of the input excitation energy is injected to its first mode. Therefore, in many practical situations, the only dominant mode in the dynamics of dynamical systems is its first mode. Assuming this to be the case for the system being considered, one may employ the Galerkin projection method [4]. Accordingly, the response of the system to an initial disturbance can be assumed to be as follows:

$$w(x,t) = \frac{\varphi(x)}{\varphi(1)}w(t) \quad (19)$$

Here, $w(t)$ is the transverse displacement of the beam tip. Furthermore, $\varphi(x)$ is the first linear un-damped transversal vibrational mode of the system. $\varphi(x)$ can be used as the basis function for describing the non-linear behavior of the system. For a beam with a tip mass, $\varphi(x)$ is given by [5]:

$$\varphi(x) = \{\cos(\beta x) - \cosh(\beta x)\} - \frac{\cos(\beta) + \cosh(\beta)}{\sin(\beta) + \sinh(\beta)}\{\sin(\beta x) - \sinh(\beta x)\} \quad (20)$$

In this equation, β is the absolute value of the smallest positive root of equation (21).

$$1+\frac{1}{\cos(\beta)\cosh(\beta)}-\frac{M}{m}\beta\left\{\tan(\beta)-\tanh(\beta)\right\}=0 \quad (21)$$

Substituting equation (19) into equation (13), multiplying it by $\varphi(x)$ and then integrating the resulting equation over the dimensionless domain, the following non-linear ordinary differential equation is obtained.

$$\frac{d^2w(t)}{dt^2}+(c_2/c_1)\frac{dw(t)}{dt}+(c_3/c_1)w(t) +(c_4/c_1)w(t)^3+(c_5/c_1)w(t)u(t)=0 \quad (22)$$

Furthermore, by substituting equation (19) into equation (14), the following equation is obtained for the axial displacement of the beam tip.

$$\frac{d^2u(t)}{dt^2}+(1/d_1)u(t)+(d_2/d_1)w(t)^2=0 \quad (23)$$

In equations (22) and (23), c_i's $1\le i\le 5$ and d_j's $1\le j\le 4$ are defined as follows.

$$c_1=\int_0^1\left(\varphi(x)^2+\sigma_2\varphi(x)^2\,\tilde{f}(1)\right)dx \quad (24)$$

$$c_2=C_t\int_0^1\varphi(x)^2\,dx \quad (25)$$

$$c_3=\int_0^1\varphi(x)\frac{d^4\varphi(x)}{dx^4}dx \quad (26)$$

$$c_4=-\frac{\sigma_1}{2\varphi(1)^2}\int_0^1\left(\frac{d\varphi(x)}{dx}\right)^2dx \times\int_0^1\varphi(x)\frac{d^2\varphi(x)}{dx^2}dx \quad (27)$$

$$c_5=-\sigma_1\int_0^1\varphi(x)\frac{d^2\varphi(x)}{dx^2}dx \quad (28)$$

$$d_1=\lambda_1 \quad (29)$$

$$d_2=\frac{1}{2\varphi(1)^2}\int_0^1\left(\frac{d\varphi(x)}{dx}\right)^2dx \quad (30)$$

In order for the coefficients of equations (22) and (23) to appear at the same order, the following dimensionless variable is introduced.

$$\tau=t\big/\sqrt{d_1} \quad (31)$$

Substituting equation (31) into equations (22) and (23), the following equations are obtained.

$$\frac{d^2w(\tau)}{d\tau^2}+\omega_n^2.w(\tau)+C_1\frac{dw(\tau)}{d\tau}+C_2w(\tau)^3 +C_3w(\tau)u(\tau)=0 \quad (32)$$

$$\frac{d^2u(\tau)}{d\tau^2}+u(\tau)+D_1w(\tau)^2=0 \quad (33)$$

where ω_n, C_i's ($1\le i\le 3$) and D_j's ($1\le j\le 2$) are defined as

$$\omega_n=\sqrt{\frac{c_3d_1}{c_1}} \quad (34)$$

$$C_1=\frac{c_2}{c_1}\sqrt{d_1} \quad (35)$$

$$C_2=\frac{c_4d_1}{c_1} \quad (36)$$

$$C_3=\frac{c_5d_1}{c_1} \quad (37)$$

$$D_1=d_2 \quad (38)$$

The natural frequency ω_n in equation (32) is not the actual frequency but instead a normalized one.

3. SOLUTION PROCEDURE

A beam with a tip mass with characteristics given in Table 1 is considered.

Table 1: Characteristics of the simulated beam and its tip mass

symbol	definition	value
E	Young's Modulus of elasticity of the beam material	69 GPa
ρ	Density of the beam material	$7800\ \mathrm{kg/m^3}$
l	Beam's length	0.15 m
b	Beam's width	0.015 m
h	Beam's thickness	0.001 m
M	Tip mass	0.050 kg

In order to give the reader some insight into the order of magnitude of all intermediate parameters defined in the paper, their values are compiled Table 2.

Table 2: Values of the intermediate parameters defined in the analysis. Parameters listed without any units represent normalized quantities.

$A=1.5\times10^{-5}m^2$	$c_3=13.729$
$I=1.25\times10^{-12}m^4$	$c_4=8.845\times10^5$
$m=1.755\times10^{-2}$ kg	$c_5=1.477\times10^6$
$\sigma_1=2.7\times10^5$	$d_1=1.055\times10^{-5}$
$\sigma_2=2.849$	$d_2=0.598$
$\lambda_1=1.055\times10^{-5}$	$\omega_n=3.334\times10^{-3}$
$\beta=0.993$	$C_2=0.716$
$c_1=13.033$	$C_3=1.196$
$D_1=0.598$	

In addition, the damping coefficient C_t is selected such that the final damping coefficients become C_1 = 0.001. For a practical choice of dimensions, as listed in Table 1, even though the non-normalized natural frequency is finite (equal to 37.6 rad/s), the normalized natural frequency ω_n in equation (32) is very small. This is simply a consequence of the fact that time is normalized via equation (31) using the natural frequency of the axial direction dynamics, given by equation (23).

From equations (6) and (7), it can be mathematically proven that

$$D_1 = C_2 / C_3 \quad (38)$$

This relation is corroborated by the numerical values of D_1, C_2, and C_3 listed in Table 2. By studying the beam constraint model, Awtar and Sen [13] showed that the above-mentioned equivalence is an invariant characteristic and independent of the beam's dimensions given in Table 1. Therefore, using equation (38) and (33), equation (32) may be rewritten as

$$\frac{d^2 w(\tau)}{d\tau^2} + \omega_n^2 . w(\tau) + C_1 \frac{dw(\tau)}{d\tau} - C_3 w(\tau) \frac{d^2 u(\tau)}{d\tau^2} = 0 \quad (38)$$

In Figure 3, the numerical solution of equation (38) and that of its corresponding linear form are compared for the initial conditions *w(0)* = 0.1 and *u(0)* = – 0.006. It is observed that these two solutions, which represent the transverse dynamics, do not differ appreciably. However, the non-linear terms in equation (33) considerably affect the axial dynamics.

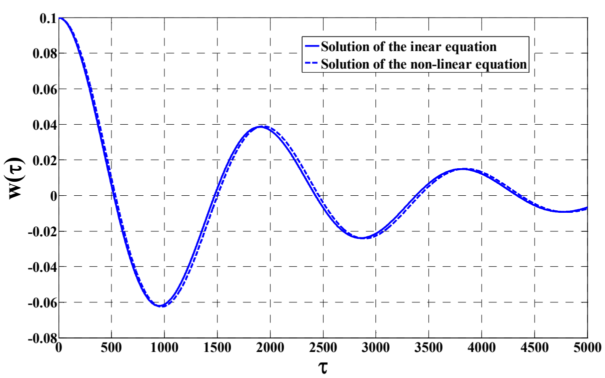

Figure 3 *Comparison of the solutions of equation (38) and its corresponding linear form*

In the next section, HPM is used in parallel with the multiple time scale perturbation method to solve the non-linear system of ODE's given in equations (32) and (33).

3-2. SOLUTION PROCEDURE

Now, HPM is utilized in parallel with the multiple time scale perturbation method to derive analytical closed form solutions for equations (32) and (33). To do so, the homotopy forms $\Im_i(P, w(\tau), u(\tau))$ for *i = 1* and 2 are constructed as

$$\Im_1(P, w(\tau), u(\tau)) = \frac{d^2 w(\tau)}{d\tau^2} + \omega_n^2 . w(\tau) + P.C_1 \frac{dw(\tau)}{d\tau} + P^2 \left(C_2 w(\tau)^3 + C_3 w(\tau) u(\tau) \right) = 0 \quad (39)$$

$$\Im_2(P, w(\tau), u(\tau)) = \frac{d^2 u(\tau)}{d\tau^2} + u(\tau) + P D_1 w(\tau)^2 = 0 \quad (40)$$

At this step, the independent variable τ is expanded in terms of multiple time scales $T_0 = \tau$ and $T_1 = P\tau$ so that the first and the second time derivatives become

$$\frac{d}{d\tau}(\ ) = \frac{\partial}{\partial T_0}(\ ) + P \frac{\partial}{\partial T_1}(\ ) \quad (41)$$

$$\frac{d^2}{d\tau^2}(\ ) = \frac{\partial^2}{\partial T_0^2}(\ ) + 2P \frac{\partial^2}{\partial T_0 \partial T_1}(\ ) + P^2 \frac{\partial^2}{\partial T_1^2}(\ ) \quad (42)$$

The solution of equations (39) and (40) are sought in the form

$$w(T_0, T_1) = w_0(T_0, T_1) + P.w_1(T_0, T_1) \quad (43)$$

$$u(T_0, T_1) = u_0(T_0, T_1) + P u_1(T_0, T_1) \quad (44)$$

By substituting equations (41), (42), (43) and (44) into the homotopy forms and equating the coefficients of like powers of P, the following equations are obtained.

P^0 :

$$\frac{\partial^2 w_0(T_0, T_1)}{\partial T_0^2} + \omega_n^2 w_0(T_0, T_1) = 0 \quad (45)$$

$$\frac{\partial^2 u_0(T_0, T_1)}{\partial T_0^2} + u_0(T_0, T_1) = 0 \quad (46)$$

P^1 :

$$\frac{\partial^2 w_1(T_0, T_1)}{\partial T_0^2} + \omega_n^2 w_1(T_0, T_1) = -2 \frac{\partial^2 w_0(T_0, T_1)}{\partial T_1 \partial T_0} - C_1 \frac{\partial w_0(T_0, T_1)}{\partial T_0} \quad (47)$$

$$\frac{\partial^2 u_1(T_0, T_1)}{\partial T_0^2} + u_1(T_0, T_1) = -2 \frac{\partial^2 u_0(T_0, T_1)}{\partial T_0 \partial T_1} - D_1 w_0(T_0, T_1)^2 \quad (48)$$

Equations (45) and (46) constitute a system of linear ordinary differential equations with constant coefficients and their solution can be written as:

$$w_0(T_0, T_1) = A_1(T_1) \exp(I\omega T_0) + \bar{A}_1(T_1) \exp(-I\omega T_0) \quad (49)$$

$$u_0(T_0, T_1) = B_1(T_1) \exp(I T_0) + \bar{B}_1(T_1) \exp(-I T_0) \quad (50)$$

where $A_1(T_1)$ and $B_1(T_1)$ are complex functions and $\bar{A}_1(T_1)$ and $\bar{B}_1(T_1)$ are the complex conjugate of $A_1(T_1)$ and $B_1(T_1)$ respectively.

Substituting equations (49) and (50) into equations (47) and (48), these equations can be solved easily using the theory of linear ODEs. But any particular solution of equations (47) and (48), contains a secular term, $T_0 exp(\pm I\omega_0 T_0)$ and $T_0 exp(\pm IT_0)$ unless the coefficients of $T_0 exp(I\omega T_0)$ and $T_0 exp(IT_0)$ in the right hand side of equations (47) and (48) are zero respectively. Therefore the following equations have to be satisfied in order to avoid any secular term in the response.

$$2I\omega_n \frac{dA_1(T_1)}{dT_1} + I\omega_n C_1 A_1(T_1) = 0 \qquad (51)$$

$$2I\frac{dB_1(T_1)}{dT_1} = 0 \qquad (52)$$

For solving equations (51) and (52), it is convenient to write $A_1(T_1)$ and $B_1(T_1)$ in the form

$$A_1(T_1) = \frac{1}{2}a_1(T_1)\exp(\alpha_1(T_1)I) \qquad (53)$$

$$B_1(T_1) = \frac{1}{2}b_1(T_1)\exp(\beta_1(T_1)I) \qquad (54)$$

By substituting equations (53) and (54) into equations (51) and (52), making the necessary simplifications and equating both the real and imaginary parts of these equations with zero, the following equations are obtained.

$$\frac{da_1(T_1)}{dT_1} + \frac{1}{2}C_1 a_1(T_1) = 0 \qquad (55)$$

$$\frac{d\alpha_1(T_1)}{dT_1} = 0 \qquad (56)$$

$$\frac{db_1(T_1)}{dT_1} = 0 \qquad (57)$$

$$b_1\frac{d\beta_1(T_1)}{dT_1} = 0 \qquad (58)$$

Equations (55) to (58) can be solved consequently and the results would be as follows.

$$a_1(T_1) = a_1\exp\left(-\frac{1}{2}C_1 T_1\right) \qquad (59)$$

$$\alpha_1(T_1) = \alpha_1 \qquad (60)$$

$$b_1(T_1) = b_1 \qquad (61)$$

$$\beta_1(T_1) = \beta_1 \qquad (62)$$

Using equations (49), (50), (53), (54), (59), (60), (61) and (62), $w_0(T_0,T_1)$ and $u_0(T_0,T_1)$ are obtained as follows.

$$w_0(T_0,T_1) = a_1\exp\left(-\frac{1}{2}C_1 T_1\right)\cos(\omega_n T_0 + \alpha_1) \qquad (63)$$

$$u_0(T_0,T_1) = b_1\cos(T_0 + \beta_1) \qquad (64)$$

As it will be seen later, a zero order approximation is sufficient for predicting the time domain behavior of $w(\tau)$, but for accurate prediction of the behavior of $u(\tau)$ at least a first order perturbation approximation is required. So, by substituting equations (63) and (64) into equation (48) and solving the resulting equation, $u_1(T_0,T_1)$ is obtained as equation (65).

$$u_1(T_0,T_1) = \frac{-D_1 a_1^2\exp(-C_1 T_1)}{2} * \left(1 + \frac{\cos(2\omega_n T_0 + 2\alpha_1)}{1-4\omega^2}\right) \qquad (65)$$

By substituting $T_0 = \tau$ into equation (63) and substituting equations (64), (65), $P = 1$, $T_0 = \tau$ and $T_1 = \tau$ into equation (44), the zero order and first order approximate solutions for $w(\tau)$ and $u(\tau)$ respectively are obtained as follows.

$$w(\tau) = a_1\exp\left(-\frac{1}{2}C_1\tau\right)\cos(\omega_n\tau + \alpha_1) \qquad (66)$$

$$u(\tau) = b_1\cos(\tau + \beta_1) + \frac{-D_1 a_1^2\exp(-C_1\tau)}{2} * \left(1 + \frac{\cos(2\omega_n\tau + 2\alpha_1)}{1-4\omega^2}\right) \qquad (67)$$

Figure 4 and Figure 5 compares the result of the presented analytical model with the numerical simulation for an un-damped and a damped system respectively. It is observed that when there is no internal resonance in the system, the analytical results well follow the numerical ones and as a result, the presented analysis can be used to investigate the dynamics of flexure beams in flexure mechanisms.

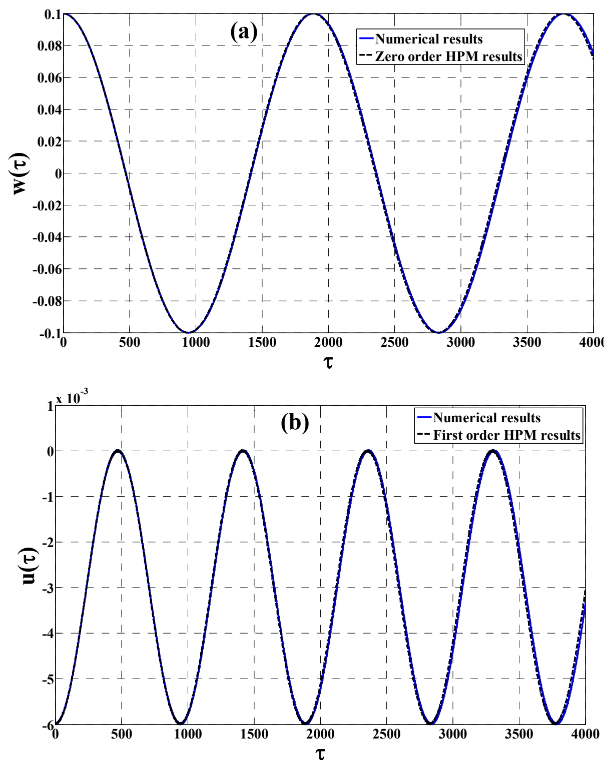

Figure 4 *Comparison of analytical results with numerical simulations for an un-damped system with initial conditions w(0) =0.1 and u(0)= – 0.006. (a) Normalized transverse displacement w(τ), and (b) Normalized axial displacement u(τ)*

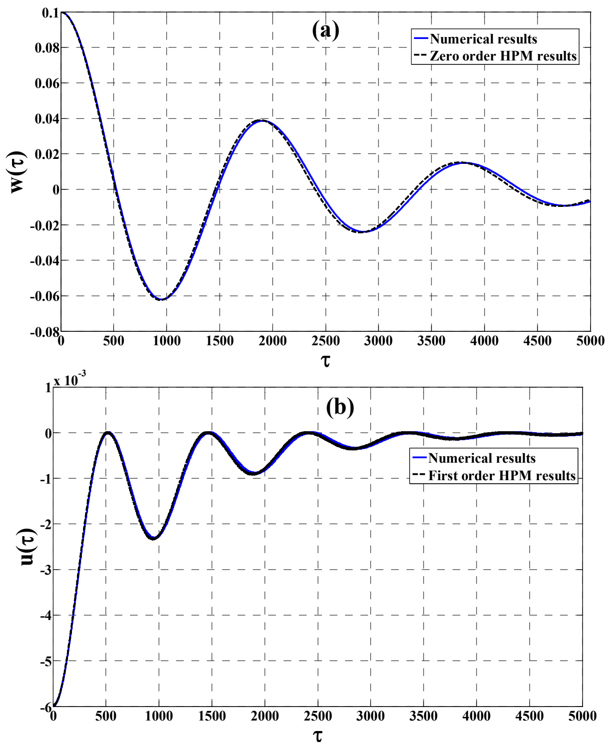

Figure 5: *Comparison of the analytical results with numerical simulations for a damped system with C_1 = 0.001 and initial conditions w(0) =0.1 and u(0)= – 0.006. (a) Normalized transverse displacement w(τ), and (b) Normalized axial displacement u(τ)*

In Figure 4 (b) and Figure 5 (b), the normalized axial displacement is composed of a large-amplitude, low-frequency component and a small-amplitude, high-frequency component. The former is due to the effect of the transverse vibration of the beam on its axial vibration while the latter is the direct consequence of the large axial stiffness of the beam. Zoomed views of the latter component are shown in Figure 6. At higher values of normalized time, the difference between the results is due to a slight difference between the frequency of the numerical and analytical solutions.

The case $\omega_n \approx 1/2$

Next, one may mathematically analyze the case when ω_n is close to 1/2, which represents a condition of internal resonance in the system of non-linear ordinary differential equations given by (32) and (33). It is important to note that this normalized value of ω_n actually corresponds to a natural frequency of 5874 rad/sec. At such large frequencies, the approximations made in deriving equations (32) and (33) break down. To accurately analyze the dynamics of the system in this frequency range, several transverse and axial modes will need to be considered and the axial kinetic energy cannot be ignored. Therefore, solving the above equations for the case when $\omega_n \approx 1/2$ is a strictly mathematical exercise and of little physical relevance. Nevertheless, a closed-form solution is presented here for the sake of completeness.

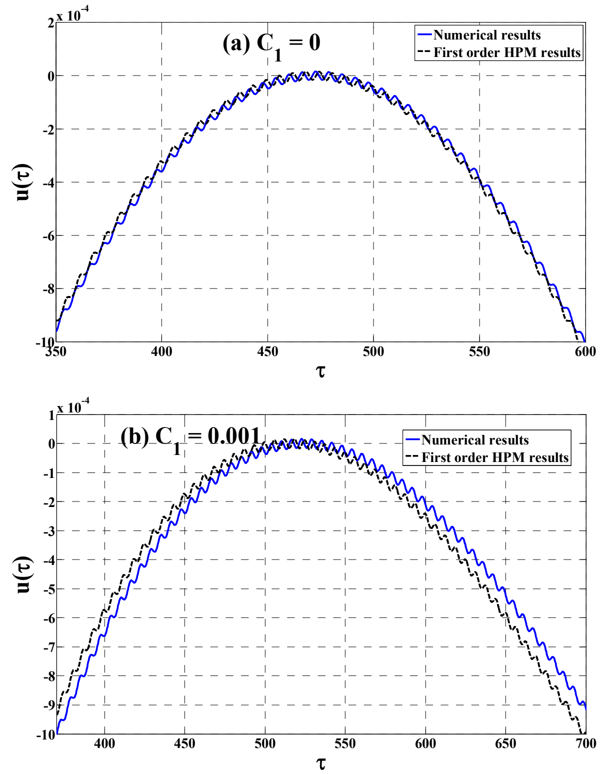

Figure 6*: Zoomed view of the normalized axial displacement (a) Un-damped system, and (b) Damped system, C_1 = 0.001*

In the case $\omega_n \approx 1/2$, the nearness of ω_n to 1/2 can be expressed as follows.

$$\omega_n = \frac{1}{2} + P\sigma \tag{68}$$

which leads to

$$\exp\left(\pm I\omega_n T_0\right) = \exp\left(\pm\frac{IT_0}{2}\right)\exp\left(\pm I\sigma T_1\right) \tag{69}$$

$$\exp\left(\pm IT_0\right) = \exp\left(\pm 2I\omega_n T_0\right)\exp\left(\mp 2I\sigma T_1\right) \tag{70}$$

By using equations (49) and (50) and substituting equations (69) and (70) into the right hand side of equations (47) and (48) respectively, the terms capable of producing secular terms are obtained as

$$2I\omega_n \frac{dA_1\left(T_1\right)}{dT_1} + IC_1 A_1\left(T_1\right)\omega_n = 0 \tag{71}$$

$$2I\frac{dB_1\left(T_1\right)}{dT_1} + D_1 A_1\left(T_1\right)^2 \exp\left(2I\sigma T_1\right) = 0 \tag{72}$$

Substituting equations (53) and (54) into equations (71) and (72) and equating the real and imaginary parts of both equations with zero gives

$$\omega_n \frac{da_1\left(T_1\right)}{dT_1} + \frac{1}{2}\omega_n C_1 a_1\left(T_1\right) = 0 \tag{73}$$

$$\frac{d\alpha_1(T_1)}{dT_1}=0 \tag{74}$$

$$\frac{db_1(T_1)}{dT_1}+\frac{1}{4}D_1a_1(T_1)^2\sin\left(2\alpha_1(T_1)\right. \\ \left.-\beta_1(T_1)+2\sigma T_1\right)=0 \tag{75}$$

$$-b_1(T_1)\frac{d\beta_1(T_1)}{dT_1}+\frac{1}{4}D_1a_1(T_1)^2\cos\left(2\alpha_1(T_1)\right. \\ \left.-\beta_1(T_1)+2\sigma T_1\right)=0 \tag{76}$$

Equations (73) to (76) can be transformed into an autonomous system by letting

$$\gamma_1(T_1)=2\alpha_1(T_1)-\beta_1(T_1)+2\sigma T_1 \tag{77}$$

The results are

$$\frac{da_1(T_1)}{dT_1}=-\frac{1}{2}C_1a_1(T_1) \tag{78}$$

$$\frac{d\gamma_1(T_1)}{dT_1}=2\sigma-\frac{d\beta(T_1)}{dT_1} \tag{79}$$

$$\frac{db_1(T_1)}{dT_1}=-\frac{1}{4}D_1a_1(T_1)^2\sin\left(\gamma(T_1)\right) \tag{80}$$

$$b_1(T_1)\frac{d\beta_1(T_1)}{dT_1}=\frac{1}{4}D_1a_1(T_1)^2\cos\left(\gamma(T_1)\right) \tag{81}$$

By solving equations (78) to (81), one can find $a_1(T_1)$, $\gamma_1(T_1)$, $b_1(T_1)$ and $\beta_1(T_1)$.

After eliminating secular terms, the solution of equations (47) and (48) becomes

$$w_1(T_0,T_1)=0 \tag{82}$$

$$u_1(T_0,T_1)=-\frac{1}{2}D_1a_1(T_1)^2 \tag{83}$$

Using equations (63), (64), (82) and (83), the final solution for $w(\tau)$ and $u(\tau)$ would be

$$w(\tau)=a_1\exp\left(-\frac{1}{2}C_1\tau_1\right)\cos\left(\omega_nT_0+\alpha_1\right) \tag{84}$$

$$u(\tau)=-\frac{1}{2}D_2a_1^2\exp\left(-C_1\tau_1\right)+b_1(\tau)\cos\left(\tau+\beta_1(\tau)\right) \tag{85}$$

4. CONCLUSION

The importance of analytically studying the dynamics of flexure mechanisms to better inform their design and optimization is well-recognized. However, such an investigation is complicated by the fact that in many applications, geometric non-linearities in flexure mechanics play an important role in the dynamics of the system. As a starting point in a broader investigation, we considered a simple beam flexure with a tip mass in this paper and analyzed its large amplitude in-plane oscillations. Euler beam approximation was made while retaining the geometric non-linearity associated with beam arc-length conservation, resulting in a set of coupled non-linear partial differential equations of dynamics. The Galerkin's projection approach was employed to reduce these to coupled non-linear ordinary differential equations, using the first transverse mode of the corresponding linearized system. To solve these equations, the homotopy perturbation method was utilized in conjunction with the method of multiple time scales. These equations reveal an internal resonance state in the system arising from the coupling between the transverse and axial directions. The time domain response of the system around and far from this internal resonance state was investigated and analytical zero-order and first order expressions for the beam tip displacement have been presented. Comparison of the derived analytical results with the numerical simulations was used to validate the accuracy of the presented closed-form solution approach. While the approximations and assumptions made in this solution approach are justified by physical and mathematical arguments, the final results are yet to be validated via experimental measurements. In addition to experimental validation, our on-ongoing research effort includes extending the above analysis approach to more complex flexure modules and mechanisms.

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