Distribution of Real and Imaginary Zeros of Multi-DoF Undamped Flexible Systems

Siddharth Rath, Arunav Maheshwari, and Shorya Awtar
Precision Systems Design Lab, Mechanical Engineering
University of Michigan, Ann Arbor, MI 48103, USA

Abstract – **This paper investigates the distribution of zeros with respect to the poles on the imaginary and real axes of the s-plane in the transfer function of a multi-DoF undamped flexible LTI system. The transfer function of a multi-DoF undamped flexible LTI system can be modally decomposed i.e. expressed as the sum of second order modes where each mode is characterized by two system parameters – modal residue and modal frequency. It is well known that when all the modal residue signs are the same, all the zeros of the multi-DoF undamped flexible LTI system are minimum phase (MP). However, the same sign for all modal residues is a sufficient condition for the elimination of non-minimum phase (NMP) zeros and not a necessary one. In order to find sufficient conditions for the elimination of NMP zeros when all modal residue signs are not the same, specific results are needed that explain the distribution of zeros with respect to the poles in the s-plane. Therefore, in this paper results are provided that elucidate the distribution of zeros with respect to the poles on the real and imaginary axes of the s-plane for any combination of modal residue signs. The real and imaginary axes are divided into four segments based on the location of the poles and the parity of the number of zeros (i.e. even or odd) in each segment is derived as a function of the system parameters. The results in this paper provide the necessary and sufficient condition for the occurrence of pole-zero flipping on the imaginary axis which is known to be detrimental to the closed-loop dynamic performance of undamped flexible LTI systems. The results from this paper will also enable the derivation of a sufficient condition for the elimination of NMP zeros in multi-DoF undamped flexible LTI systems.**

I. INTRODUCTION AND MOTIVATION

Flexible systems are widely used in motion and vibration control applications such as space structures [1, 2], rotorcraft blades [3, 4], hard-disk drives [5, 6], flexure mechanisms [7-9], and motion systems with transmission compliance [10, 11], among others. They often require the use of feedforward and feedback control in order to achieve desirable dynamic performance which is characterized by high speed, low settling time, strong disturbance rejection, stability robustness, etc. The distribution of the zeros with respect to the poles of the transfer function in the s-plane plays a critical role in determining the achievable dynamic performance of multi-DoF undamped flexible LTI systems.

There are different ways in which distribution of the zeros with respect to the poles can change as a function of the parameters of the multi-DoF undamped flexible LTI system. The marginally minimum phase (MMP) zero pair i.e. a complex conjugate zero pair that lies purely on the imaginary axis can transition into non-minimum phase (NMP) zeros as illustrated in Fig.1a and Fig.1b. The presence of non-minimum phase zeros limits the dynamic performance that can be achieved through feedback and feedforward control strategies [12-14]. The MMP zero pair can remain on the imaginary axis but its position with respect to the pole pairs can flip as illustrated in Fig.1c which can lead to a 180° phase loss near the frequency range where the pole-zero flip occurs. This phenomenon is called pole-zero flipping [15]. The pole-zero flipping has no effect on the open loop stability of the plant i.e. multi-DoF undamped flexible LTI system but can destabilize a given combination of controller and plant in closed loop [16].

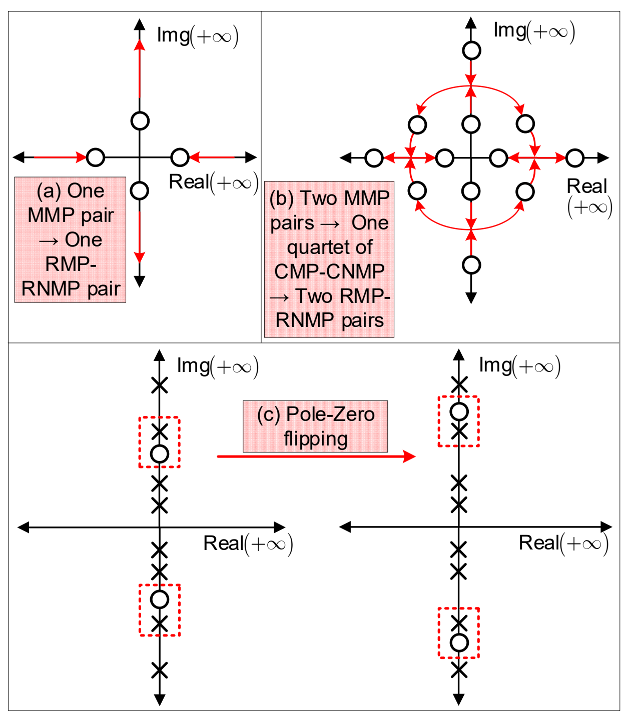

Figure 1. Different distributions of zeros with respect to the poles

The transfer function of a multi-DoF undamped flexible LTI system can be modally decomposed into second order modes where each mode is characterized by two system parameters – modal residue and modal frequency [17]. Martin [18] showed that when all the modal residue signs are the same, all the zeros of the multi-DoF undamped flexible LTI system are minimum phase (MP). It was further shown that this elimination of NMP zeros is robust to the variations in system parameters caused by modelling uncertainties and/or unmodelled dynamics [18-21] as long all the modal residue signs remain the same. Collocated actuator-sensor configuration is one technique that guarantees the same sign for all modal residues [16, 22]. However, certain trajectory tracking applications such as tracking the tip displacement of a flexible link robot while providing input torque at its root requires a non-collocated actuator-sensor configuration [23]. This non-collocated actuator sensor configuration can lead to the occurrence of NMP zeros in the transfer functions of multi-DoF undamped flexible LTI systems [24]. Different researchers proposed different linear combinations of outputs from multiple sensors in order to achieve the same sign for all modal residues. This technique guaranteed the elimination of NMP zeros in multi-DoF undamped flexible LTI systems with non-collocated actuator sensor configurations [23, 25, 26]. In all these studies, the focuses were the investigation of minimum phase zeros obtained when all modal residue signs are the same and the investigation of actuator-sensor placements in order to achieve the same sign for all modal residues. However, the same sign for all modal residues is only a sufficient condition for the elimination of NMP zeros from the transfer functions of multi-DoF undamped flexible LTI systems [27]. It may not always be possible to achieve this condition given various constraints on the number and location of actuators and sensors [7]. Hence, it is important to investigate the distribution of zeros with respect to the poles when all the modal residue signs are not the same and derive sufficient conditions in terms of the system parameters to eliminate NMP zeros.

The zero dynamics of multi-DoF undamped flexible LTI systems when all modal residue signs are not the same has been less thoroughly studied analytically. However, there are several numerical studies for specific multi-DoF undamped flexible LTI systems [7, 16, 28-31]. These flexible LTI systems employ a single actuator and sensor in non-collocated configurations which lead to all modal residue signs not being the same. The transfer functions of certain flexure mechanism based motion stages [7, 28, 29] have demonstrated the transition of two pairs of marginally minimum phase (MMP) zeros into a quartet of complex minimum phase (CMP) – complex non-minimum phase (CNMP) zeros, as illustrated in Fig. 1b, for small changes in the mass distribution and operating position of the motion stages. Canon demonstrated that the transfer function of a multi-DoF undamped flexible LTI system consisting of rigid disks attached serially via flexible rods [16] undergoes pole-zero flipping as illustrated in Fig. 1c. It was shown that a small variation in the mass of one of the rigid disks led to this pole-zero flipping. Spector [30] and Lee [31] carried out numerical investigation of the transfer function of a pinned-free beam model and free-free beam model respectively and identified the migration of the zeros on the real and imaginary axis due to variation in sensor position.

In all these studies, the focus is the numerical investigation of the distribution of the zeros with the respect to the poles and the presence of NMP zeros in specific multi-DoF undamped flexible LTI systems. They do not provide any broader conclusions from the numerical results. Rath [27] carried out analytical investigation of the zeros of a three-DoF undamped flexible LTI system by constructing zero loci that comprehensively covered all possible distribution of the zeros with respect to the poles for any value of system parameters. Based on these zero loci, the necessary and sufficient conditions for the elimination of NMP zeros were provided. However, this work was limited to only three-DoF undamped flexible LTI systems. It is easier to construct all possible zero loci that span the entire parameter space and use them to derive the necessary and sufficient conditions for the elimination of NMP zeros for low-DoF undamped flexible LTI systems such as three-DoF. However, this process becomes tedious and complicated as the number of DoFs increase. Therefore, in this paper instead of providing the zero loci for all possible distribution of zeros with respect to the poles for a multi-DoF undamped flexible LTI system, we provide results on the parity of the number of zeros with respect to the poles on the real and imaginary axis as a function of system parameters (*Section III*). Based on these results, *Section III* also provides the necessary and sufficient condition for pole-zero flipping to occur on the imaginary axis as shown in Fig. 1c.

The results in *Section III* also enable the derivation of a non-unique sufficient condition for the elimination of NMP zeros from the transfer functions of multi-DoF undamped flexible LTI systems when all modal residue signs are not the same. This sufficient condition will be covered in a future research paper. *Section II* provides the underlying assumptions for this paper. *Section IV* provides the conclusion and briefly discusses the course of future research.

II. ZERO DYNAMICS AND MODAL DECOMPOSITION

The input-output dynamics of a LTI system given by transfer function $G(s)$ can be expressed as the sum of the contributions of its decomposed modes where the coefficients a_i and b_i are real numbers.

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = \sum_{i=1}^{n} \frac{\alpha_i}{s + p_i} \qquad (1)$$

Assumption 1: The LTI flexible system investigated in this paper is assumed such that all the decomposed modes are second order, and that there are no first order modes. Additionally, it is assumed that these second order modes are all oscillatory in nature (i.e. the poles associated with each mode lie on the imaginary axis). This is a reasonable assumption for many continuous structural and discrete spring-mass systems [7, 29, 32].

Assumption 2: Next, it is assumed that the flexible system is undamped. This assumption is reasonable for many flexible systems such as flexure mechanisms with no rolling or sliding joints and negligible damping from the air [7, 29] and for space structures [1, 2, 33] where damping is negligible.

Assumption 3: If force is assumed to be the input and displacement is selected as the output of such a flexible LTI system, then the input-output transfer function $G(s)$ from (1) can be restated as follows:

$$G(s)=\frac{N(s)}{D(s)}=\frac{b_m s^{2m}+\cdots+b_1 s^2+b_0}{a_n s^{2n}+\cdots+a_1 s^2+a_0}=\sum_{i=1}^{n}\frac{\alpha_i}{s^2+\omega_i^{\,2}} \quad (2)$$

Furthermore, it is assumed that $\alpha_i \neq 0$ for any i from 1 to n and $\omega_1 < \omega_2 < \ldots < \omega_n$ without any loss of generality. Here the total number of second order modes is n, which is also the DoF of the system. ω_i and α_i are the modal frequency and the modal residues of the i^{th} mode, respectively, and are called the system parameters per the nomenclature of this paper. Additionally, it is assumed that $G(s)$ represents a physical system (as opposed to a mathematical system), and is strictly proper (i.e. $m < n$). In other words, the number of zero pairs is less than the number of modes or pole pairs in the system and $2n$-$2m$ is the relative degree of the transfer function $G(s)$. Since, the coefficients of the numerator $N(s)$, given by b_i are real, the zeros occur in complex conjugate pairs. The numerator $N(s)$ is also an even function hence the distribution of the zeros of the transfer function $G(s)$ is symmetric about the real and imaginary axis. Therefore, it is sufficient to focus on the zeros lying on the positive real axis (from origin to $+\infty$) and the positive imaginary axis (from origin to $+\infty$).

III. PARITY OF NUMBER OF ZEROS ON THE REAL AND IMAGINARY AXIS

The transfer function of a multi-DoF undamped flexible LTI system that follows **Assumptions 1**, **2**, and **3** is denoted by $G(s)$ and mathematically expressed by (2). The distribution of the zeros of the transfer function $G(s)$ on the imaginary and real axis of the s-plane is given in terms of the parity of the number of the zeros i.e. odd or even number of zeros between two points, $s = c_1$ and $s = c_2$, either both on the real axis or both on the imaginary axis with $|c_1| < |c_2|$. The choice of c_1 and c_2 divides the imaginary and the real axis into four distinct segments for which the parity of the number of zeros is investigated. These segments are graphically illustrated in Fig. 2 and described below:

1. Segment 1: Parity of number of zeros between $c_1 = j\omega_f$ and $c_2 = j\omega_{f+1}$ on the imaginary axis for any f from 1 to $n-1$.
2. Segment 2: Parity of number of zeros between $c_1 = j\omega_n$ (pole corresponding to the last mode of $G(s)$) and $c_2 = +j\infty$ on the imaginary axis.
3. Segment 3: Parity of number of zeros between c_1 = origin and $c_2 = +\infty$ on the real axis.
4. Segment 4: Parity of number of zeros between c_1 = origin and $c_2 = j\omega_1$ (pole corresponding to the first mode of $G(s)$) on the imaginary axis.

When seeking to find the parity of the number of zeros between two points, c_1 and c_2, the sign of $N(s)$, given by (2), will be sought at both points. Since $N(s)$ is a continuous function in s, examining its sign at points c_1 and c_2 tells us how many times $N(s)$ can become zero between these two points. An odd number of zeros occur between points c_1 and c_2 if the sign of $N(c_1)N(c_2) < 0$ and an even number of zeros occur between points c_1 and c_2 if the sign of $N(c_1)N(c_2) > 0$. The opposite statements are also true i.e. if there are odd number of zeros between points c_1 and c_2 then $N(c_1)N(c_2) < 0$ and if there are even number of zeros between points c_1 and c_2 then $N(c_1)N(c_2) > 0$. For the case where $N(c_1)N(c_2) = 0$ because $N(c_1) = 0$ and/or $N(c_2) = 0$, they will be replaced by their respective limits as shown in (3). If $N(c_2)\rightarrow\infty$, then it will replaced by its limiting case as shown in (3).

$$\begin{aligned}&\text{If } N(c_1)=0 \Rightarrow \text{ Replace } N(c_1) \text{ with } \lim_{s\to c_1^+}\operatorname{sgn}(N(s))\\ &\text{If } N(c_2)=0 \text{ or} \rightarrow \infty \Rightarrow \text{ Replace } N(c_2) \text{ with } \lim_{s\to c_2^-}\operatorname{sgn}(N(s))\end{aligned} \quad (3)$$

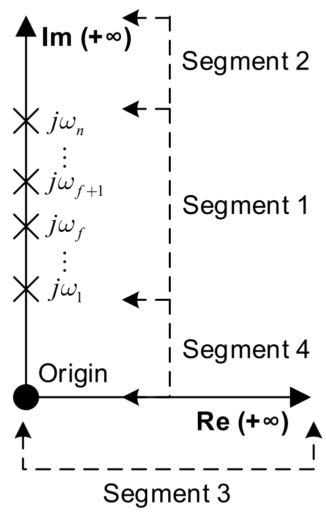

Figure 2. Imaginary and Real axis divided into four distinct segments

A. Parity of number of zeros between $j\omega_f$ and $j\omega_{f+1}$

Result 1: In a multi-DoF undamped flexible LTI system whose transfer function is given by (2), the parity of number of zeros between any two of its consecutive poles, given by $j\omega_f$ and $j\omega_{f+1}$ for any f from 1 to $n-1$, is given by (4).

$$\begin{aligned}&\operatorname{sgn}\left\{\left(\alpha_f\right)\left(\alpha_{f+1}\right)\right\}>0 \Leftrightarrow \text{Odd no. of zeros between } j\omega_f \text{ and } j\omega_{f+1}\\ &\operatorname{sgn}\left\{\left(\alpha_f\right)\left(\alpha_{f+1}\right)\right\}<0 \Leftrightarrow \text{Even no. of zeros between } j\omega_f \text{ and } j\omega_{f+1}\end{aligned} \quad (4)$$

Proof: Express the transfer function $G(s)$ as the sum of three transfer functions.

$$
\begin{aligned}
&G(s) = G_{f,f+1}(s) + \frac{\alpha_f}{s^2+\omega_f^2} + \frac{\alpha_{f+1}}{s^2+\omega_{f+1}^2} \\
&\text{where } G_{f,f+1}(s) = \frac{N_{f,f+1}(s)}{D_{f,f+1}(s)} = \sum_{\substack{i=1\\ i\neq f,f+1}}^{n}\left(\frac{\alpha_i}{s^2+\omega_i^2}\right)
\end{aligned}
\tag{5}
$$

Next, the numerator of $G(s)$ which given by $N(s)$ is evaluated at $s = j\omega_f$ and $s = j\omega_{f+1}$.

$$
\begin{aligned}
&\Rightarrow N(s) = N_{f,f+1}(s)\left(s^2+\omega_f^2\right)\left(s^2+\omega_{f+1}^2\right) \\
&\qquad + D_{f,f+1}(s)\left[\alpha_f\left(s^2+\omega_{f+1}^2\right)+\alpha_{f+1}\left(s^2+\omega_f^2\right)\right] \\
&\Rightarrow N(j\omega_f) = D_{f,f+1}(j\omega_f)\left[\alpha_f\left(\omega_{f+1}^2-\omega_f^2\right)\right] \\
&\Rightarrow N(j\omega_{f+1}) = -D_{f,f+1}(j\omega_{f+1})\left[\alpha_{f+1}\left(\omega_{f+1}^2-\omega_f^2\right)\right]
\end{aligned}
\tag{6}
$$

Taking the product of $N(j\omega_f)$ and $N(j\omega_{f+1})$ and evaluating the sign of the product leads to (7).

$$
\begin{aligned}
&N(j\omega_f)N(j\omega_{f+1}) = -\left(\alpha_f\alpha_{f+1}\right)\left(D_{f,f+1}(j\omega_f)D_{f,f+1}(j\omega_{f+1})\right) \\
&\qquad \left(\omega_{f+1}^2-\omega_f^2\right)^2 \\
&\Rightarrow \operatorname{sgn}\left(N(j\omega_f)N(j\omega_{f+1})\right) = -\operatorname{sgn}\left(\alpha_f\alpha_{f+1}\right) \\
&\qquad \operatorname{sgn}\left(D_{f,f+1}(j\omega_f)D_{f,f+1}(j\omega_{f+1})\right)
\end{aligned}
\tag{7}
$$

Observe that

$$
D_{f,f+1}(s) = \prod_{i=1, i\neq f,f+1}^{n}\left(s^2+\omega_i^2\right) \tag{8}
$$

$$
\begin{aligned}
\Rightarrow \operatorname{sgn}\left(D_{f,f+1}(j\omega_f)\right) &= \operatorname{sgn}\left(D_{f,f+1}(j\omega_{f+1})\right) \\
&= \prod_{i=1}^{f-1}(-1)\prod_{i=f+2}^{n}(1)
\end{aligned}
$$

Substituting (8) in (7), we get (4) which is restated below.

$$
\operatorname{sgn}\left(N(j\omega_f)N(j\omega_{f+1})\right) = -\operatorname{sgn}\left(\alpha_f\alpha_{f+1}\right)
$$

Therefore,

$$
\begin{aligned}
&\operatorname{sgn}\left(\alpha_f\alpha_{f+1}\right) < 0 \Leftrightarrow \operatorname{sgn}\left(N(j\omega_f)N(j\omega_{f+1})\right) > 0 \Leftrightarrow \text{Even} \\
&\qquad \text{no. of zeros between } j\omega_f \text{ and } j\omega_{f+1} \\
&\operatorname{sgn}\left(\alpha_f\alpha_{f+1}\right) > 0 \Leftrightarrow \operatorname{sgn}\left(N(j\omega_f)N(j\omega_{f+1})\right) < 0 \Leftrightarrow \text{Odd} \\
&\qquad \text{no. of zeros between } j\omega_f \text{ and } j\omega_{f+1}
\end{aligned}
$$

Result 1 provides the necessary and sufficient condition for pole-zero flipping to occur on the imaginary axis. Assume that a single MMP zero pair flips its position with respect to a pole pair as the system parameters are varied and there are no other pole-zero flippings at that instance. Therefore, a pole-zero flipping at pole $j\omega_f$ will only occur when the parity of number of zeros between poles $j\omega_{f-1}$ and $j\omega_f$ and between poles $j\omega_f$ and $j\omega_{f+1}$ changes. **Result 1** proves that the change in the parity of number of zeros in these two sub-segments will occur only if the sgn($\alpha_f\,\alpha_{f-1}$) and sgn($\alpha_f\,\alpha_{f+1}$) undergo a change in sign. Since, it has been assumed that no other pole-zero flipping occurs at the same instance, the parity of number of zeros between other neighboring poles remains the same and hence their corresponding modal residue signs remain the same due to **Result 1**.

Therefore, the only way that sgn($\alpha_f\,\alpha_{f-1}$) and sgn($\alpha_f\,\alpha_{f+1}$) undergo a change in sign is if α_f undergoes a change in sign. Hence, pole-zero flipping on the imaginary axis occurs if and only if the modal residue corresponding to the pole (i.e. modal frequency) undergoes a change in sign.

As an example, consider Fig.1c which illustrates the pole-zero flipping as observed in [16]. It can be assumed without any loss of generality that the modal residue corresponding to the first pole is positive. Based on the distribution of poles and zeros on the imaginary axis in Fig. 1c and by using **Result 1**, it can be inferred that before the pole-zero flipping, the modal residue sign sequence corresponding to the four modes are (+ − − +). After the pole zero flipping, the modal residue sign sequence becomes (+ − + +). Therefore, it can be seen that the change in the sign of the modal residue corresponding to the third mode is responsible for the pole-zero flipping. This information can be used to choose physical parameters of the flexible system so that under small parametric variations, the modal residue corresponding to the third mode does not switch signs thus avoiding pole-zero flipping.

B. Parity of number of zeros between $j\omega_n$ and $j\infty$

Result 2: In a multi-DoF undamped flexible LTI system whose transfer function is given by (2), the parity of number of zeros between the last pole, given by $j\omega_n$, and positive infinity on the imaginary axis, $j\infty$, is given by (9)

$$
\begin{aligned}
&(-1)^{n-1+m}\operatorname{sgn}\left\{(b_m)(\alpha_n)\right\} < 0 \Leftrightarrow \text{Odd no. of zeros} \\
&\qquad \text{between } j\omega_n \text{ and } j\infty \\
&(-1)^{n-1+m}\operatorname{sgn}\left\{(b_m)(\alpha_n)\right\} > 0 \Leftrightarrow \text{Even no. of zeros} \\
&\qquad \text{between } j\omega_n \text{ and } j\infty \\
&\text{where} \\
&b_m = \sum_{i=1}^{n}\alpha_i e_{n-1-m}\left(\omega_1,\omega_2,\ldots,\omega_{i-1},\omega_{i+1}\ldots,\omega_{n-1},\omega_n\right) \\
&\text{where } m\in(0,n-1) \text{ is the greatest positive integer} \\
&\text{such that } b_m \neq 0 \text{ and } e_{n-1-m} \text{ represents the} \\
&(n-1-m)\text{th elementary symmetric polynomial} \\
&\text{in } n-1 \text{ variables.}
\end{aligned}
\tag{9}
$$

Proof: Express the transfer function $G(s)$ as the sum of two transfer functions.

$$
\begin{aligned}
&G(s) = G_n(s) + \frac{\alpha_n}{s^2+\omega_n^2} \\
&\text{where } G_n(s) = \frac{N_n(s)}{D_n(s)} = \sum_{i=1}^{n-1}\frac{\alpha_i}{s^2+\omega_i^2}
\end{aligned}
\tag{10}
$$

Next, the sign of the numerator of $G(s)$ which is given by $N(s)$ is evaluated at $s = j\omega_n$ as shown in (11).

$$
\begin{aligned}
&N(s) = N_n(s)\left(s^2+\omega_n^2\right) + \alpha_n D_n(s) \\
&\Rightarrow N(j\omega_n) = \alpha_n D_n(j\omega_n) \\
&\Rightarrow \operatorname{sgn}\left(N(j\omega_n)\right) = \operatorname{sgn}(\alpha_n)\operatorname{sgn}\left(D_n(j\omega_n)\right)
\end{aligned}
\tag{11}
$$

Observe that

$$D_n\left(j\omega_n\right)=\prod_{i=1}^{n-1}\left(\omega_i^2-\omega_n^2\right)$$
$$\Rightarrow \operatorname{sgn}\left(D_n\left(j\omega_n\right)\right)=\prod_{i=1}^{n-1}\operatorname{sgn}\left(\omega_i^2-\omega_n^2\right)=\prod_{i=1}^{n-1}\left(-1\right) \tag{12}$$
$$=\left(-1\right)^{n-1}$$

Substituting (12) into (11) yields

$$\operatorname{sgn}\left(N\left(j\omega_n\right)\right)=\left(-1\right)^{n-1}\operatorname{sgn}\left(\alpha_n\right) \tag{13}$$

Now the sign of $N(s)$ is evaluated at $s = j\infty$. It is impossible to directly evaluate the sign of $N(s)$ at $s = j\infty$, because $N(s) \to \infty$. Therefore, the limit of $N(s)$ is considered as below, by making a change of variables and substituting for c_2 in (3).

$$\lim_{y\to\infty}\operatorname{sgn}\left(N\left(jy\right)\right)=\lim_{y\to\infty}\operatorname{sgn}\left(\sum\nolimits_{i=1}^{m}b_i\left(-1\right)^i\left(y\right)^{2i}\right) \tag{14}$$

As y tends to infinity, the term with the highest power of y in (14) which is y^{2m} will dominate. This leads to (15).

$$\lim_{y\to\infty}\operatorname{sgn}\left(N\left(jy\right)\right)=\left(-1\right)^m\operatorname{sgn}\left(b_m\right) \tag{15}$$

Using Vieta's formulae and binomial expansion, an expression for b_m can be derived in terms of the elementary symmetric polynomials, which are defined as follows.

$$\begin{aligned}&e_0\left(x_1,x_2,...,x_n\right)=1,\ e_1\left(x_1,x_2,...,x_n\right)=\sum_{1\le i\le n}x_i\\&e_2\left(x_1,x_2,...,x_n\right)=\sum_{1\le i<k\le n}x_ix_k \text{ and so forth, until}\\&e_n\left(x_1,x_2,...,x_n\right)=\prod_{1\le i\le n}x_i\end{aligned} \tag{16}$$

The expression for b_m is stated in (17).

$$b_m=\sum\nolimits_{i=1}^{n}\alpha_ie_{n-1-m}(\omega_1,\omega_2,...,\omega_{i-1},\omega_{i+1}...,\omega_{n-1},\omega_n) \tag{17}$$

Taking the product of the signs of $N(s)$ at $s = j\omega_n$ and $s = j\infty$ using (13) and (15) gives (9), restated below:

$$\operatorname{sgn}\left(N\left(j\omega_n\right)\right)\lim_{s\to j\infty}\operatorname{sgn}\left(N\left(s\right)\right)=\left(-1\right)^{n-1+m}\operatorname{sgn}\left\{\left(b_m\right)\left(\alpha_n\right)\right\}$$

Therefore,

$$\left(-1\right)^{n-1+m}\operatorname{sgn}\left\{\left(b_m\right)\left(\alpha_n\right)\right\}<0\Leftrightarrow\text{Odd no. of zeros between } j\omega_n \text{ and } j\infty$$
$$\left(-1\right)^{n-1+m}\operatorname{sgn}\left\{\left(b_m\right)\left(\alpha_n\right)\right\}>0\Leftrightarrow\text{Even no. of zeros between } j\omega_n \text{ and } j\infty$$

The term $2(n-m)$ is the relative degree of the transfer function of $G(s)$ and b_m corresponds to the highest power of s in the numerator as shown in (2). There are classes of multi-DoF undamped flexible LTI systems whose relative degree depends only on actuator and sensor location and is independent of the other physical parameters such as mass and stiffness distribution [32, 34]. In such cases, **Result 2** can prove to be quite useful in evaluating the effect of mass and stiffness distribution on the parity of number of zeros in Segment 2 for a given actuator and sensor location. A change in the parity of number of zeros in Segment 2 indicates that odd number of zeros may have migrated (i) To/From Segment 1 from/to Segment 2 which suggests pole-zero flipping close to the frequency ω_n (ii) To/From Segment 3 from/to Segment 2 leading to transition from MMP zeros to real non-minimum phase (RNMP) zeros or vice-versa.

C. Parity of number of zeros between 0 *and* ∞ *on the real axis*

Result 3: In a multi-DoF undamped flexible LTI system whose transfer function is given by (2), the parity of number of zeros between the origin and positive infinity on the real axis is given by (18)

$$\begin{aligned}&\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(b_m\right)\right\}<0\Leftrightarrow\text{Odd no. of zeros between 0 and } +\infty\\&\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(b_m\right)\right\}>0\Leftrightarrow\text{Even no. of zeros between 0 and } +\infty\end{aligned} \tag{18}$$

where $b_{q^*}=\sum\nolimits_{i=1}^{n}\alpha_ie_{n-1-q^*}\left(\omega_1,\omega_2,...,\omega_{i-1},\omega_{i+1}...,\omega_{n-1},\omega_n\right)$
where $q^*\in\left(0,n-1\right)$ is the smallest positive integer such that $b_{q^*}\neq 0$ and e_{n-1-q^*} represents the $\left(n-1-q^*\right)$th elementary symmetric polynomial in $n-1$ variables.

Proof: The sign of $N(s)$ is evaluated at $s = \infty$.

$$\lim_{x\to\infty}\operatorname{sgn}\left(N\left(x\right)\right)=\lim_{x\to\infty}\operatorname{sgn}\left(\sum\nolimits_{i=1}^{m}b_ix^{2i}\right) \tag{19}$$

As x tends to infinity, the term with the highest power of x in (19) which is x^{2m} will dominate. This leads to (20).

$$\lim_{x\to\infty}\operatorname{sgn}\left(N\left(x\right)\right)=\operatorname{sgn}\left(b_m\right) \tag{20}$$

Equation (17) provides the expression for b_m. We evaluate the sign of $N(s)$ at the origin by substituting $c_1 = 0$ into (3).

$$\lim_{x\to 0^+}\operatorname{sgn}\left(N\left(x\right)\right)=\lim_{x\to 0}\operatorname{sgn}\left(\sum\nolimits_{i=0}^{m}b_ix^{2i}\right) \tag{21}$$

Observe that

$$x^k\gg x^{k+1}\text{ as }x\to 0 \tag{22}$$

Hence, the term with the lowest power of x in (21) which is x^{2q^*} will dominate due to (22).

$$\lim_{x\to 0^+}\operatorname{sgn}\left(N\left(x\right)\right)=\operatorname{sgn}\left(b_{q^*}\right) \tag{23}$$

Note that an expression for this coefficient can be obtained by substituting q^* for m in (17).

$$b_{q^*}=\sum\nolimits_{i=1}^{n}\alpha_ie_{n-1-q^*}(\omega_1,\omega_2,...,\omega_{i-1},\omega_{i+1}...,\omega_{n-1},\omega_n) \tag{24}$$

Taking the product of the sign or sign limits at 0 and ∞ using (20) and (23) yields (18), restated below:

$$\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(b_m\right)\right\}<0\Leftrightarrow\text{Odd no. of zeros between 0 and }\infty$$
$$\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(b_m\right)\right\}>0\Leftrightarrow\text{Even no. of zeros between 0 and }\infty$$

There is a class of multi-DoF undamped flexible LTI system for which $q^* = 0$ for any actuator and sensor location and mass stiffness distribution [32]. A change in the parity of number of zeros in Segment 3 indicates that odd number of zeros may have migrated (i) From/to Segment 2 to/from Segment 3 leading to transition from MMP to RNMP zeros or vice-versa (ii) From/to Segment 4 to/from Segment 3 leading to transition from MMP to RNMP zeros or vice-versa.

D. Parity of number of zeros between 0 *and* $j\omega_1$ *on the imaginary axis*

Result 4: In a multi-DoF undamped flexible LTI system whose transfer function is given by (2), the parity of number of zeros between the origin and the first pole, given by $j\omega_1$ on the imaginary axis is given by (25).

$$\begin{aligned}(-1)^{q^*}\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(\alpha_1\right)\right\} < 0 \Leftrightarrow \text{Odd no. of zeros} \\ \text{between } j0 \text{ and } j\omega_1 \\ (-1)^{q^*}\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(\alpha_1\right)\right\} > 0 \Leftrightarrow \text{Even no. of zeros} \\ \text{between } j0 \text{ and } j\omega_1\end{aligned} \tag{25}$$

Proof: Express the transfer function $G(s)$ as the sum of two transfer functions.

$$\begin{aligned}G(s) &= G_1(s) + \frac{\alpha_1}{s^2+\omega_1^2} \\ \text{where } G_1(s) &= \frac{N_1(s)}{D_1(s)} = \sum\nolimits_{i=2}^{n}\frac{\alpha_i}{s^2+\omega_i^2}\end{aligned} \tag{26}$$

Next, the sign of the numerator of $G(s)$ which is given by $N(s)$ is evaluated at $s = j\omega_1$ as shown in (27).

$$\begin{aligned}&N(s) = N_1(s)\left(s^2+\omega_1^2\right) + \alpha_1 D_1(s) \\ &\Rightarrow N(j\omega_1) = \alpha_1 D_1(j\omega_1) \\ &\Rightarrow \operatorname{sgn}\left(N(j\omega_1)\right) = \operatorname{sgn}(\alpha_1)\operatorname{sgn}\left(D_1(j\omega_1)\right)\end{aligned} \tag{27}$$

Observe that

$$\begin{aligned}&D_1(j\omega_1) = \prod\nolimits_{i=2}^{n}\left(\omega_i^2-\omega_1^2\right) \\ &\Rightarrow \operatorname{sgn}\left(D_1(j\omega_1)\right) = \prod\nolimits_{i=2}^{n}(1) = 1\end{aligned} \tag{28}$$

Substituting (28) into (27) yields

$$\operatorname{sgn}\left(N(j\omega_1)\right) = \operatorname{sgn}(\alpha_1) \tag{29}$$

Now the sign of $N(s)$ will be evaluated at $j0$, by substituting for c_1 in (3)

$$\begin{aligned}&\lim_{y\to 0^+}\operatorname{sgn}\left(N(jy)\right) = \lim_{y\to 0^+}\operatorname{sgn}\left(\sum\nolimits_{i=0}^{m} b_i (jy)^{2i}\right) \\ &= \lim_{y\to 0^+}\operatorname{sgn}\left(\sum\nolimits_{i=0}^{m}(-1)^i b_i y^{2i}\right)\end{aligned} \tag{30}$$

Based on the procedure followed in (21) and (22) in *Section III-C*, (30) can then be simplified to (31) as shown below.

$$\lim_{y\to 0^+}\operatorname{sgn}\left(N(jy)\right) = (-1)^{q^*}\operatorname{sgn}\left(b_{q^*}\right) \tag{31}$$

Taking the product of the sign or sign limits at $j0$ and $j\omega_1$ using (29) and (31) yields (25), restated below:

$$\begin{aligned}(-1)^{q^*}\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(\alpha_1\right)\right\} < 0 \Leftrightarrow \text{Odd no. of zeros} \\ \text{between } j0 \text{ and } j\omega_1 \\ (-1)^{q^*}\operatorname{sgn}\left\{\left(b_{q^*}\right)\left(\alpha_1\right)\right\} > 0 \Leftrightarrow \text{Even no. of zeros} \\ \text{between } j0 \text{ and } j\omega_1\end{aligned}$$

A change in the parity of number of zeros in Segment 4 indicates that odd number of zeros may have migrated (i) To/From Segment 1 from/to Segment 4 which suggests pole-zero flipping close to the frequency ω_1 (ii) To/From Segment 3 from/to Segment 4 leading to transition from MMP to RNMP zeros or vice-versa.

IV. CONCLUSION AND FUTURE RESEARCH

This paper investigates the distribution of zeros with respect to the poles on the imaginary and real axes of the s-plane for a multi-DoF undamped flexible LTI system. The investigation is carried out by dividing the imaginary and real axis into four distinct segments and deriving the parity of number of zeros in each segment as a function of system parameters – modal residues and modal frequencies. **Result 1** leads to the necessary and sufficient condition for pole-zero flipping to occur on the imaginary axis. It is found that pole-zero flipping on the imaginary axis occurs if and only if the modal residue corresponding to the pole (i.e. modal frequency) undergoes a change in sign. This necessary and sufficient condition will enable informed physical design choices like stiffness and mass distribution and actuator and sensor placement in order to guarantee the elimination of pole-zero flipping that tend to destabilize the controller and plant dynamics in closed loop. **Results 2**, **3** and **4** provide certain mathematical conditions for the transition of odd number of MMP zero pairs to odd number of RMP – RNMP zero pairs and vice-versa.

Results 1, **2**, **3** and **4** also enable the derivation of a sufficient condition for the elimination of NMP zeros in multi-DoF undamped flexible LTI systems. The derivation of this sufficient condition will be covered in a future research paper. However, the steps for the derivation are discussed here briefly. **Result 1** enables the derivation of a sufficient condition in terms of the sequence of modal residue signs that guarantee the elimination of only CNMP zeros from the transfer functions of multi-DoF undamped flexible LTI systems. Considering the derived sufficient condition for the elimination of only CNMP zeros holds true, **Results 2**, **3**, and **4** enable the derivation of the necessary and sufficient condition for the elimination of RNMP zeros, therefore leading to a sufficient condition that guarantee the elimination of all types of NMP zeros. It will be shown via numerical simulations in the future papers that this sufficient condition will enable informed physical design choices like stiffness and mass distribution and actuator and sensor placement. These choices will guarantee that the elimination of all types of NMP zeros which are detrimental to the dynamic performance of multi-DoF undamped flexible LTI systems.

REFERENCES

1. Gennaro, S.D., 2003, "Output stabilization of flexible spacecraft with active vibration suppression," IEEE Transactions on Aerospace and Electronic Systems, **39**(3), p. 747-759.
2. Hu, Q., 2008, "Sliding mode maneuvering control and active vibration damping of three-axis stabilized flexible spacecraft with actuator dynamics," Nonlinear Dynamics, **52**(3), p. 227-248.
3. Friedmann, P.P. and Millott, T.A., 1995, "Vibration reduction in rotorcraft using active control - A comparison of various approaches," Journal of Guidance, Control, and Dynamics, **18**(4), p. 664-673.
4. Giurgiutiu, V., 2000, "Review of Smart-Materials Actuation Solutions for Aeroelastic and Vibration Control," Journal of Intelligent Material Systems and Structures, **11**(7), p. 525-544.
5. Chang, J.Y., 2007, "Hard disk drive seek-arrival vibration reduction with parametric damped flexible printed circuits," Microsystem Technologies, **13**(8), p. 1103-1106.

6. Feng, G., Fook Fah, Y., and Ying, Y., 2005, "Modeling of hard disk drives for vibration analysis using a flexible multibody dynamics formulation," IEEE Transactions on Magnetics, **41**(2), p. 744-749.
7. Awtar, S. and Parmar, G., 2013, "Design of a Large Range XY Nanopositioning System," Journal of Mechanisms and Robotics, **5**(2), p. p.021008.
8. Roy, N.K. and Cullinan, M.A., 2018, "Design and characterization of a two-axis, flexure-based nanopositioning stage with 50 mm travel and reduced higher order modes," Precision Engineering, **53**, p. 236-247.
9. Roy, N.K. and Cullinan, M.A., 2018, "Fast Trajectory Tracking of a Flexure-Based, Multiaxis Nanopositioner With 50-mm Travel," IEEE/ASME Transactions on Mechatronics, **23**(6), p. 2805-2813.
10. Chalhoub, N.G. and Ulsoy, A.G., 1986, "Dynamic Simulation of a Leadscrew Driven Flexible Robot Arm and Controller," Journal of Dynamic Systems, Measurement, and Control, **108**(2), p. 119-126.
11. Varanasi, K.K. and Nayfeh, S.A., 2004, "The Dynamics of Lead-Screw Drives: Low-Order Modeling and Experiments," Journal of Dynamic Systems, Measurement, and Control, **126**(2), p. 388-396.
12. Freudenberg, J. and Looze, D., 1985, "Right half plane poles and zeros and design tradeoffs in feedback systems," IEEE Transactions on Automatic Control, **30**(6), p. 555-565.
13. Middleton, R.H., 1991, "Trade-offs in linear control system design," Automatica, **27**(2), p. 281-292.
14. Hoagg, J.B. and Bernstein, D.S., 2007, "Nonminimum-phase zeros-much to do about nothing-classical control-revisited part II," IEEE Control Systems Magazine, **27**(3), p. 45-57.
15. Rosenthal, D.E., 1984, "Experiments in Control of Flexible Structures with Uncertain Parameters," PhD Dissertation, Stanford University, Stanford, CA.
16. Cannon, R.H. and Rosenthal, D.E., 1984, "Experiments in control of flexible structures with noncolocated sensors and actuators," Journal of Guidance, Control, and Dynamics, **7**(5), p. 546-553.
17. Meirovitch, L., 1967, *Analytical methods in Vibration*, Vol. 19, The Macmillan Company, New York.
18. Martin, G.D., 1978, "On the control of flexible mechanical systems," Ph.D. dissertation, Stanford University, Stanford, CA.
19. Williams, T., 1992, "Model order effects on the transmission zeros of flexible space structures," Journal of guidance, control, and dynamics, **15**(2), p. 540-543.
20. Williams, T. and Juang, J.N., 1992, "Sensitivity of the transmission zeros of flexible space structures," Journal of guidance, control, and dynamics, **15**(2), p. 368-375.
21. Fleming, F.M., 1990, "The effect of structure, actuator, and sensor on the zeroes of controlled structures," Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA.
22. Burke, S.E., Hubbard, J.E., and Meyer, J.E., 1993, "Distributed transducers and colocation," Mechanical Systems and Signal Processing, **7**(4), p. 349-361.
23. Saad, M., Saydy, L., and Akhrif, O., 2000, "Noncollocated passive transfer functions for a flexible link robot," CACSD. Conference Proceedings. IEEE International Symposium on Computer-Aided Control System Design (Cat. No.00TH8537), p. 971-975.
24. Cannon, R.H. and Schmitz, E., 1984, "Initial Experiments on the End-Point Control of a Flexible One-Link Robot," The International Journal of Robotics Research, **3**(3), p. 62-75.
25. Liu, L.-Y. and Yuan, K., 2003, "Noncollocated passivity-based PD control of a single-link flexible manipulator," Robotica, **21**(2), p. 117-135.
26. Wang, D. and Vidyasagar, M., 1990, "Passive control of a single flexible link," Proceedings., IEEE International Conference on Robotics and Automation, p. 1432-1437.
27. Rath, S., Cui, L., and Awtar, S., 2021, "On the Zeros of an Undamped Three-DOF Flexible System," ASME Letters in Dynamic Systems and Control, **1**(4), p. 041010.
28. Cui, L., Okwudire, C., and Awtar, S., 2017, "Modeling complex nonminimum phase zeros in flexure mechanisms," Journal of Dynamic Systems, Measurement, and Control, **139**(10), p. 101001.
29. Cui, L. and Awtar, S., 2019, "Experimental validation of complex non-minimum phase zeros in a flexure mechanism," Precision Engineering, **60**, p. 167-177.
30. Spector, V.A. and Flashner, H., 1989, "Sensitivity of Structural Models for Noncollocated Control Systems," Journal of Dynamic Systems, Measurement, and Control, **111**(4), p. 646-655.
31. Lee, Y.J. and Speyer, J.L., 1993, "Zero locus of a beam with varying sensor and actuator locations," Journal of guidance, control, and dynamics, **16**(1), p. 21-25.
32. Hoagg, J.B., Chandrasekar, J., and Bernstein, D.S., 2006, "On the Zeros, Initial Undershoot, and Relative Degree of Collinear Lumped-Parameter Structures," Journal of Dynamic Systems, Measurement, and Control, **129**(4), p. 493-502.
33. Hughes, P.C., 1974, "Dynamics of flexible space vehicles with active attitude control," Celestial mechanics, **9**(1), p. 21-39.
34. Wang, D. and Vidyasagar, M., 1991, "Transfer Functions for a Single Flexible Link," The International Journal of Robotics Research, **10**(5), p. 540-549.