Non-minimum Phase Zeros of Two-DoF Damped Flexible Systems

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Abstract – This paper presents an investigation of the non-minimum phase (NMP) zeros in the SISO dynamics of a two-DoF damped flexible LTI system. The presence of undamped poles and non-minimum phase (NMP) zeros limits the performance of feedforward and feedback control in dynamic systems. While, the addition of passive viscous damping is beneficial for the undamped poles of the flexible systems, it can lead to the presence of NMP zeros in the system dynamics. Using modal decomposition, it is shown in this paper that as long as all the modal residue signs of multi-DoF damped flexible LTI system are same, the addition of passive viscous damping does not lead to NMP zeros in the system dynamics for any value of system parameters – modal residue, modal frequency and modal damping ratio. Further, the zero loci of a two-DoF damped flexible LTI system is investigated in order to understand how the addition of passive viscous damping can lead to NMP zeros when the modal residue signs are different. Based on this zero loci, the necessary and sufficient conditions are derived that guarantee the elimination of all NMP zeros for a two-DoF damped flexible LTI system when the modal residue signs are different.

Keywords – complex non-minimum phase, real non-minimum phase, viscous damping, modal decomposition, zero loci, classical damping

1. INTRODUCTION AND BACKGROUND

Flexible systems are relevant to motion and vibration control applications such as space structures (Hu 2008), rotorcraft blades (Friedmann and Millott 1995, Giurgiutiu 2000), hard-disk drives (Chang 2007, Feng et al. 2005), flexure mechanisms (Awtar and Parmar 2013, Choi and Lee 2005), and motion systems with transmission compliance (Chalhoub and Ulsoy 1986, Varanasi and Nayfeh 2004), among others. They often require the use of feedforward and feedback control in order to achieve desirable dynamic performance, generally characterized by high speed, low settling time, good disturbance rejection, and stability robustness. However, the presence of undamped poles and non-minimum phase (NMP) zeros in the single input single output (SISO) dynamics of the flexible LTI system lead to significant challenges in achieving the desired dynamic performance (Freudenberg and Looze 1985).The application of passive viscous damping can mitigate these challenges by moving the poles of the flexible system to the left hand side (LHS) of the imaginary axis. This can lead to the desired dynamic performance due to higher achievable closed loop bandwidth. However, in order to realize such performance, it should be made sure that the addition of viscous damping does not inadvertently lead to the appearance of NMP zeros. NMP zeros are defined as zeros that strictly lie on the right hand side (RHS) of the imaginary axis.

There have been several studies on the effect of passive viscous damping on the poles of damped flexible linear time invariant (LTI) systems, but less so on the zeros. Thompson (Thompson 1981) used the root locus technique to find the optimal natural frequency and viscous damping ratio for a two-DoF dynamic vibration absorber. Engelen (Engelen et al. 2007) derived the complex eigenvalues of a flexible structure including a viscous damper. They proposed approximate solutions for the complex eigenvalues and formulas for the maximum modal damping ratio and the optimal damping constant to minimize the residual vibration in the flexible structure. In both these studies, the desired placement of the poles on the LHS of the imaginary axis was achieved by varying the physical location and value of the viscous damper. Tsai (Tsai et al. 2009) studied the forced vibration of Timoshenko beam with distributed internal viscous damping and found that the amplitude of resonant oscillations of the beam can be reduced by tuning the viscous damping value. Varanasi (Varanasi and Nayfeh 2006) conducted analytical and experimental investigation of foam based viscous damping in flexure mechanisms and found that it can lead to the damping of multiple resonant modes in the flexible system by pushing the poles to the LHS of the imaginary axis. In all these studies, the primary focus was how the poles of continuous or discrete flexible systems change through the application of passive viscous damping. However, none of these studies investigated the impact of viscous damping on the system zeros.

In fact, there is a very limited body of research that investigates the impact of passive viscous damping on the system zeros, even though zeros also play an important role in determining the dynamic performance of the flexible systems. Hoagg (Hoagg et al. 2006) numerically demonstrated the presence of complex non-minimum phase (CNMP) zeros in the SISO dynamics of a three-DoF damped flexible LTI system for large value of damping ratio (ζ>1.3). However, they did not report any necessary or sufficient conditions in terms of the parameters of the flexible system to eliminate the CNMP zeros that were observed. Pang (Pang et al. 1993) studied the migration of open loop poles and zeros of an Euler-Bernoulli beam and showed that both lie on the LHS of the imaginary axis. However, this study was system specific and was limited to the investigation of one particular transfer function obtained by assuming collocated actuator-sensor placement. The paper provided no commentary on whether the conclusions that were reached are applicable to any general collocated transfer function. Alberts (Thomas et al. 1986) analytically studied the effect of distributed passive viscous damping on non-collocated transfer functions for transverse vibration of beams and reported the existence of NMP zeros. However, no necessary or sufficient conditions were reported on how to eliminate NMP zeros in the presence of damping.

Therefore, there remains a need for an analytical investigation to find the necessary and sufficient conditions to eliminate NMP zeros from the SISO transfer functions of general flexible systems in the presence of passive viscous damping. In order to carry out this analytical investigation, we assume that the flexible systems considered in this paper are classically damped. Classical damping has widespread application in engineering practices because of its conceptual simplicity and ease of application (Roesset et al. 1973, Tsai 1974, Thomson et al. 1974). This assumption also makes the analytical investigation in this paper mathematically amenable, as shown in *Section 2*, and therefore allows insights into the behavior of the zero dynamics in the presence of passive viscous damping. Having assumed classical damping, we first investigate the zero dynamics of multi-DoF damped flexible LTI systems and provide a sufficient condition for the elimination of NMP zeros (*Section 3*). We are able to do this because it is easier to find a sufficient condition as compared to necessary and sufficient conditions. In order to find the necessary and sufficient conditions, one needs to find all the possible sufficient conditions over the entire parameter space of the flexible system. As the DoF of the flexible system increases, the parameter space also becomes larger. Therefore, finding all possible sufficient conditions becomes tedious. Hence, in this paper, we do not attempt to find the necessary and sufficient conditions for a general multi-DoF damped flexible LTI system.

In order to derive the sufficient as well as necessary conditions for the elimination of NMP zeros, we limit our analytical investigation to a two-DoF damped flexible LTI system (*Section 4*). Since, there are no previously derived conditions for the elimination of NMP zeros in the SISO transfer function of a two-DoF damped flexible LTI system, the investigation and results of this paper are novel to the best of our knowledge. Additionally, these results also have practical relevance. In several real applications, the dynamics of multi or infinite DoF flexible systems, with or without non-linearities, are often investigated via the simplest possible model, which can often be two-DoF flexible LTI system model. Hastings et.al. (Hastings and Book 1986) approximated the dynamics of a flexible manipulator, which is infinite DoF, using only two flexible modes in order to design an observer for the flexible system. Duffour et.al. (Duffour and Woodhouse 2004) investigated the self-excited instability in brake-disc like mechanical systems in the presence of frictional contact by approximating the linearized dynamics of the non-linear flexible system using a two-DoF flexible LTI system model. Similarly, Wang (Wang et al. 2021) investigated the effect of friction on the dynamic performance of the precision motion stages with mechanical bearings using a two-DoF flexible system model. Therefore, the necessary and sufficient conditions for the elimination of NMP zeros from the SISO transfer function of two-DoF damped flexible LTI systems (*Section 4*) will find several applications in practical engineering problems.

2. ZERO DYNAMICS AND MODAL DECOMPOSITION

Consider the equation of motion of a multi-DoF viscously damped flexible LTI system as shown in Eq.(1). [*M*] denotes the mass matrix, [*C*] denotes the damping matrix, [*K*] denotes the stiffness matrix, *F* denotes the force acting on the flexible system through an input vector [*B*] and *u* denotes the displacement that is being measured and it is a linear combination of the displacement of the individual DoFs denoted by *w*. The SISO transfer function between applied force and measured displacement i.e. $u(s)/F(s)$ is given by $G(s)$.

$$\begin{aligned} &[M]_{n\times n}\ddot{w}+[C]_{n\times n}\dot{w}+[K]_{n\times n}w=[B]_{n\times 1}F \\ &u=[D]_{1\times n}w \end{aligned} \tag{1}$$

Assumption 1: The [*M*], [*C*], and [*K*] matrices satisfy the Caughey and O'Kelly criteria (Caughey and O'Kelly 1965) given by Eq.(2)

$$[C][M]^{-1}[K]=[K][M]^{-1}[C] \tag{2}$$

Multi-DoF damped flexible LTI systems that satisfy Eq.(2) are called classically damped flexible systems. The natural mode shapes of vibration (i.e. eigenvectors) of such flexible systems are real valued and identical to those of the associated undamped flexible systems. The SISO transfer function of multi-DoF classically damped flexible LTI systems can be modally decomposed i.e. the transfer function can be written as sum of several second order modes (Meirovitch 1967). Each second order mode is characterized by three real valued system parameters namely, modal frequency (ω_i), modal residue (α_i), and modal damping ratio (ζ_i). The roots of each second order mode lie on the LHS of the imaginary axis due to the presence of passive viscous damping (positive damping) in the flexible system. If a multi-DoF damped flexible LTI system does not satisfy Eq.(2) then it is called a non-classically damped flexible system and its SISO transfer function cannot be modally decomposed as described above. Therefore, **Assumption 1** allows us to carry out the modal decomposition of the SISO transfer function $G(s)$ as shown in Eq.(3)

$$G(s)=\frac{b_{2m}s^{2m}+\ldots+b_1s+b_0}{a_{2n}s^{2n}+\cdots+a_1s+a_0}=\sum_{i=1}^{n}\frac{\alpha_i}{s^2+2\zeta_i\omega_i s+\omega_i^{\,2}} \tag{3}$$

The total number of second order modes in the modal decomposition of $G(s)$ in Eq.(3) is n. In this paper, n is also defined as the DoF of the flexible system. Since the transfer function $G(s)$ is assumed to represent the SISO dynamics of a physical system, it should be proper i.e. $m < n$.

3. MULTI-DOF DAMPED FLEXIBLE LTI SYSTEMS

A multi-DoF damped flexible LTI system that follows **Assumptions 1** can be expressed by Eq.(3). This flexible system has n (>1) second order modes.

Result 1: In a multi-DoF damped flexible LTI system, a sufficient condition for the zeros of the SISO transfer function, $G(s)$ to be minimum phase (MP) is that all the modal residue signs are the same.

Proof: The zeros for $G(s)$ are found by solving Eq.(4) given below.

$$\sum_{i=1}^{n}\frac{\alpha_i}{s^2+2\zeta_i\omega_i s+\omega_i^2}=0 \qquad (4)$$

It is assumed that $x+jy$ is one of the zeros of $G(s)$ obtained by solving Eq.(4). Substitute $x+jy$ into the i^{th} mode in order to rewrite it in the Euler form as shown below.

$$\frac{\alpha_i}{s^2+2\zeta_i\omega_i s+\omega_i^2}=\frac{\alpha_i}{\beta_i e^{j\theta_i}} \qquad (5)$$

where

$$\beta_i=\left|(x^2-y^2+2\zeta_i\omega_i x+\omega_i^2)+2j(xy+\zeta_i\omega_i y)\right|$$

$$\theta_i=\tan^{-1}\left(\frac{2xy+2\zeta_i\omega_i y}{x^2-y^2+2\zeta_i\omega_i x+\omega_i^2}\right)$$

Next, substitute the Euler form of the i^{th} mode in Eq.(4).

$$\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}e^{-j\theta_i}=0\Rightarrow\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\cos(\theta_i)-j\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\sin(\theta_i)=0 \qquad (6)$$

$$\Rightarrow\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\cos(\theta_i)=0 \quad (7) \quad \text{AND} \quad \sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\sin(\theta_i)=0 \qquad (8)$$

Since all α_i are the same sign in this result, that sign can be assumed to be positive without any loss of generality. β_i is a positive quantity, by definition, as shown in Eq.(5). If it is assumed that the zero $x+jy$ does not lie on the real axis i.e. $y \neq 0$, then the $\sin(\theta_i)$ terms are not all zero. Therefore the $\sin(\theta_i)$ terms cannot all have the same sign in order to satisfy Eq.(8). The sign of $\sin(\theta_i)$ is given by the sign of the quantity $2y(x+\zeta_i\omega_i)$. Fig. 1 illustrates the sign of $2y(x+\zeta_i\omega_i)$ when the zero $x+jy$ lies in different regions of the s-plane. For a multi-DoF damped flexible system that has n modes, the s-plane can be divided into n+1 regions defined by $x = -\zeta_i\omega_i$. If the zero $x+jy$ lies in Region 1 that extends from $x = -\zeta_1\omega_1$ to x = +∞, then all the $\sin(\theta_i)$ terms will necessarily have the same sign which would not satisfy Eq.(8). Hence, any zero of $G(s)$ that does not lie on the real axis (i.e. $y \neq 0$) can never lie in Region 1. Since Region 1 includes the right hand side (RHS) of the s-plane, such zeros will always be minimum phase. However, the zero $x+jy$ can lie in Region 2 or for that matter in any region other than Region 1 and Region n+1, since the $\sin(\theta_i)$ terms do not have the same sign in these regions, as shown in Fig. 1, and therefore Eq.(8) can be satisfied.

If it is assumed that the zero $x+jy$ lies on the real axis i.e. y = 0 and as a consequence, all the $\sin(\theta_i)$ terms are equal to zero, then the above mentioned argument does not hold anymore for the $\sin(\theta_i)$ terms. In that case, one can observe the signs of the $\cos(\theta_i)$ terms in Eq.(7). The sign of $\cos(\theta_i)$ is given by the sign of the quantity $(x^2 + 2\zeta_i\omega_i x + \omega_i^2)$ when y = 0. If the zero x lies on the RHS of the imaginary axis i.e. $x > 0$, then this quantity is always positive. As a consequence, all the $\cos(\theta_i)$ terms will be positive and Eq.(7) will not be satisfied. This leads to a contradiction that can only be resolved if $x < 0$ i.e. any real zero lies on the LHS of the imaginary axis and therefore is minimum phase.

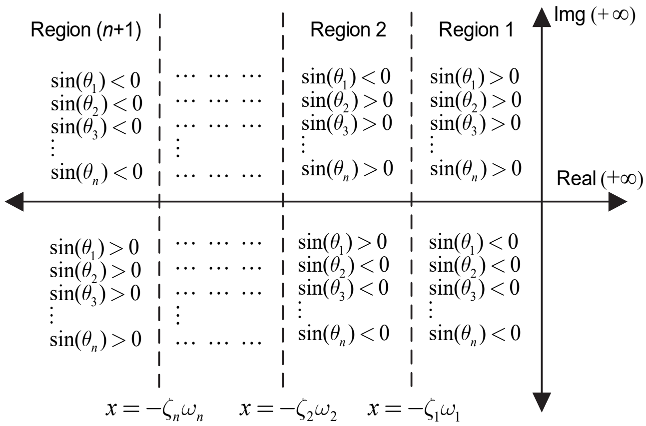

Fig. 1. Sign of $\sin(\theta_i)$ *when the zero* $(x+jy)$ *lies in various regions of the s-plane*

Result 2: In a multi-DoF damped flexible LTI system, when all the modal residues signs are the same, if all the poles are under damped, then all the zeros will also be under damped. Since zeros are necessarily minimum phase due to **Result 1**, these zero will be complex minimum phase (CMP).

Proof: Assume that a zero of G(s), given by x, lies on the real axis. This zero should satisfy Eq.(7). The sign of $\cos(\theta_i)$ is given by the sign of the quantity $(x^2 + 2\zeta_i\omega_i x + \omega_i^2)$ when y = 0. If all the poles are underdamped i.e. $\zeta_i < 1$ then the quantity $(x^2 + 2\zeta_i\omega_i x + \omega_i^2)$ will always be positive for any value of x. Therefore, all the terms in Eq.(7) will have the same sign. As a consequence, Eq.(7) cannot be satisfied for any value of x. This leads to a contradiction which can only be resolved if the zero does not lie on the real axis. Therefore, if all the poles of the damped flexible system are underdamped, the zeros will also be necessarily underdamped. Hence, all the zeros of a multi-DoF damped flexible system with same signs of modal residues and all underdamped poles will be complex minimum phase (CMP).

In this result, it has been proven that when all the modal residue signs are the same (e.g. all positive), the zeros dynamics of a multi-DoF damped flexible system is necessarily minimum phase for any value of system parameters. This is a more general result compared to a previous well-known result for multi-DoF undamped flexible systems (Martin 1978). There, it was shown that when all the modal residue signs are the same, the zeros are necessarily minimum phase and strictly lie on the imaginary axis for any value of system parameters – modal residues and modal frequencies. In the present paper, **Result 1** proves that even when viscous damping is added to such a flexible system, the zero dynamics will continue to remain minimum phase for any value of system parameters, which now additionally include modal damping ratios. Therefore, if the same sign of modal residues can be realized through physical design of a multi-

DoF damped flexible system, then it will guarantee the absence of NMP zeros. **Result 2** proves that if all the poles of the flexible system are underdamped, then the zeros will also be necessarily underdamped. **Result 1** and **Result 2** together prove that zeros of a multi-DoF damped flexible system will be necessarily complex minimum phase (CMP). These complex minimum phase zeros will lie in the region between $x = -\zeta_1\omega_1$ and $x = -\zeta_n\omega_n$, as shown in Fig.1 for any value of system parameters as long as all the poles remain underdamped and all the modal residue signs remain the same.

However, when all the modal residue signs are not same, additional conditions are required to guarantee the elimination of NMP zeros. Hence, in the next sections, the NMP zero dynamics of two-DoF damped flexible systems will be investigated to determine those additional conditions (in terms of system parameters) for the elimination of NMP zeros when all the modal residue signs are not the same.

4. TWO-DOF DAMPED FLEXIBLE LTI SYSTEM

The SISO transfer function of a two-DoF damped flexible LTI system that follows **Assumption 1** can be expressed mathematically by Eq.(9). The subscript '2' stands for the number of modes (or the DoF) in the modal decomposition. Furthermore, it is assumed without any loss of generality that $\omega_u < \omega_v$.

$$G_2(s) = \frac{\alpha_u}{s^2 + 2\zeta_u\omega_u s + \omega_u^2} + \frac{\alpha_v}{s^2 + 2\zeta_v\omega_v s + \omega_v^2} \tag{9}$$

The zeros of $G_2(s)$ are investigated by studying the roots of its numerator $N_2(s)$ as shown in Eq.(10) below. As mentioned in *Section 3*, the zeros of $G_2(s)$ will be investigated for κ (= α_u / α_v) < 0.

$$\frac{N_2(s)}{D_2(s)} = \frac{\kappa\left(s^2 + 2\zeta_v\omega_v s + \omega_v^2\right) + \left(s^2 + 2\zeta_u\omega_u s + \omega_u^2\right)}{\left(s^2 + 2\zeta_u\omega_u s + \omega_u^2\right)\left(s^2 + 2\zeta_v\omega_v s + \omega_v^2\right)}$$

$$\Rightarrow N_2(s) = \kappa A_2(s) + B_2(s) \text{ where } \kappa \triangleq \frac{\alpha_u}{\alpha_v}$$

where (10)

$$A_2(s) \triangleq s^2 + 2\zeta_v\omega_v s + \omega_v^2$$
$$B_2(s) \triangleq s^2 + 2\zeta_u\omega_u s + \omega_u^2$$

Next, we define a transfer function $T_2(s) = A_2(s) / B_2(s)$, which has no physical meaning and simply serves as a mathematical tool. The root locus of $T_2(s)$ is the zero locus of $G_2(s)$. Therefore, to obtain the zero locus of $G_2(s)$, we plot the root locus of $T_2(s)$ as a function of the ratio of modal residues (κ). Since, $\kappa < 0$ by definition in this section, the negative (i.e. complementary) root locus of $T_2(s)$ will be analyzed to derive the necessary and sufficient conditions to eliminate all NMP zeros in $G_2(s)$. In order to do so, first a sufficient condition for the elimination of CNMP zeros is derived in **Result 3** followed by the necessary and sufficient conditions for the elimination of CNMP and RNMP zeros in **Result 4**.

4.1 Sufficient Condition for Eliminating CNMP Zeros

Result 3: In a two-DoF damped flexible LTI system, when the modal residue signs are not same ($\kappa < 0$), the ratio of modal damping ratios, χ, should satisfy the following inequality to eliminate CNMP zeros for any value of modal residues.

$$\eta \le \chi \le \frac{1}{\eta} \quad \text{where} \quad \eta \triangleq \frac{\omega_u}{\omega_v},\ \chi \triangleq \frac{\zeta_u}{\zeta_v} \tag{11}$$

Proof: The zeros of $G_2(s)$ (or equivalently, the roots of $T_2(s)$) can be obtained by solving

$$N_2(s) = \kappa A_2(s) + B_2(s) = 0 \tag{12}$$

$$\Rightarrow \kappa = \frac{B_2(s)}{A_2(s)} = -\frac{\left(s^2 + 2\zeta_u\omega_u s + \omega_u^2\right)}{\left(s^2 + 2\zeta_v\omega_v s + \omega_v^2\right)}$$

If $x + jy$ is part of the root locus of $T_2(s)$ and $\kappa < 0$, then

$$\left(\angle\psi_u\right)_{s=x+jy} - \left(\angle\psi_v\right)_{s=x+jy} = 360^\circ m \tag{13}$$

where m is an integer

$$\left(\angle\psi_u\right)_{s=x+jy} \triangleq \angle\left(s^2 + 2\zeta_u\omega_u s + \omega_u^2\right)_{s=x+jy}$$
$$\left(\angle\psi_v\right)_{s=x+jy} \triangleq \angle\left(s^2 + 2\zeta_v\omega_v s + \omega_v^2\right)_{s=x+jy}$$

Applying the tangent function to both sides of Eq.(13) and substituting Laplace variable 's' with Cartesian coordinates x and y, the root locus of $T_2(s)$ in the Cartesian form is given by

$$\tan\left(\left(\angle\psi_u\right)_{s=x+jy}\right) - \tan\left(\left(\angle\psi_v\right)_{s=x+jy}\right) = 0 \tag{14}$$

$$\Rightarrow y\left(\frac{2x + 2\zeta_u\omega_u}{x^2 - y^2 + 2\zeta_u\omega_u x + \omega_u^2} - \frac{2x + 2\zeta_v\omega_v}{x^2 - y^2 + 2\zeta_v\omega_v x + \omega_v^2}\right) = 0$$

The root locus of $T_2(s)$ starts at the roots of $A_2(s)$ when $\kappa = -\infty$. The sufficient condition that guarantees the elimination of CNMP zeros of $G_2(s)$ for any value of κ ($-\infty < \kappa < 0$) exists if the root locus of $T_2(s)$ does not cross the imaginary axis at non-zero frequencies. The points of intersection of the root locus with the imaginary axis, if they exist, can be determined by substituting $x = 0$ in Eq.(14) as follows

$$\frac{2\zeta_u\omega_u y}{-y^2 + \omega_u^2} - \frac{2\zeta_v\omega_v y}{-y^2 + \omega_v^2} = 0$$
$$\Rightarrow \left(\zeta_v\omega_v - \zeta_u\omega_u\right)y^2 + \left(\zeta_u\omega_u\omega_v^2 - \zeta_v\omega_v\omega_u^2\right) = 0 \tag{15}$$

The sufficient condition to guarantee that the root locus of $T_2(s)$ does not cross the imaginary axis is to ensure that Eq.(15) does not have any real solutions for y. This can be ensured if

$$\frac{\zeta_u\omega_u\omega_v^2 - \zeta_v\omega_v\omega_u^2}{\zeta_u\omega_u - \zeta_v\omega_v} \le 0 \quad \Rightarrow \quad \frac{(\chi - \eta)}{\left(\chi - \dfrac{1}{\eta}\right)} \le 0 \tag{16}$$

Since $\eta < 1$ by definition (i.e. $\omega_u < \omega_v$), the above condition can be reduced to Eq.(11):

$$\chi \ge \eta \text{ AND } \chi \le 1/\eta \quad \Rightarrow \quad \eta \le \chi \le 1/\eta$$

Fig. 2b provides a graphical depiction of this sufficient condition via the root locus of $T_2(s)$, or equivalently the zero locus of $G_2(s)$, as κ varies from $-\infty$ to 0. It should be noted that the three different parameter ranges of χ span all values of χ from 0 to $+\infty$, and therefore the zero loci shown in Fig.2 are a comprehensive depiction of the zeros of $G_2(s)$ for all possible values of system parameters as long as $\kappa < 0$. Note that since the zeros occur as pairs of complex conjugate, only one half of the zero loci, which is above the real axis is shown in Fig. 2. Based on **Result 3** and Fig. 2, the following observations can be made about the CNMP zeros of a two-DoF damped flexible system:

1. When the sufficient condition given by Eq.(11) is met, Fig. 2b shows that the zero locus of $G_2(s)$ never crosses the imaginary axis at non-zero frequencies and the absence of CNMP zeros is guaranteed. When this condition is not met, Figs. 2a and 2c depict the zero loci crossing the imaginary axis at non-zero frequencies.
2. The sufficient condition given by Eq.(11) for the elimination of CNMP zeros is not unique or necessary. Even when this sufficient condition is not satisfied, Figs. 2a and 2c show that there are other sufficient conditions for the elimination of CNMP zeros. For each parameter range of χ in Figs. 2a and 2c, respectively, one can guarantee the absence of CNMP zeros by selecting ranges of κ for which the zeros never lie on the RHS of the s-plane (excluding the RHS real axis). These ranges of κ are not explicitly reported here but can be found easily by solving for the intersection of the zero loci with the imaginary axis and the real axis.
3. It was shown previously (Rath et al. 2021) that CNMP zeros never occur in the dynamics of a two-DoF undamped flexible system for any value of system parameters (modal frequencies and residues). However, Figs. 2a and 2c depict the presence of CNMP zeros for certain parameter ranges of χ and κ. This shows a potential disadvantage of adding damping to a two-DoF flexible system, which is intuitively unexpected because damping is generally beneficial (Varanasi and Nayfeh 2006). Therefore, one important advantage of the sufficient condition given by Eq.(11) is that it guarantees the elimination of CNMP zeros even if the modal residues of the two-DoF damped flexible system undergo large variations.

4.2 Necessary and Sufficient Conditions for Eliminating CNMP and RNMP zeros

The sufficient condition given by Eq.(11) that guarantees the elimination of CNMP zeros does not guarantee the elimination of RNMP zeros. In fact, since $y = 0$ is always a solution of Eq.(14), the real axis will always be part of the zero locus of $G_2(s)$ for certain ranges of κ for each parameter range of χ, as can been seen graphically in Figs. 2a, 2b, and 2c. In this section, we will determine the necessary and sufficient conditions for eliminating CNMP as well as RNMP zeros.

Result 4: In a two-DoF damped flexible LTI system, when the modal residue signs are not same ($\kappa < 0$), the following conditions are individually sufficient, and together necessary, to guarantee the elimination of all NMP zeros.

$$
\begin{aligned}
&\text{Condition 4.1: } (\chi<\eta) \quad \text{AND} \quad (\kappa\le -1 \ \text{ OR } \ \kappa\ge -\eta\chi)\\
&\text{OR}\\
&\text{Condition 4.2: } (\eta\le\chi\le 1/\eta) \quad \text{AND} \quad (\kappa\le -1 \ \text{ OR } \ \kappa\ge -\eta^2)\\
&\text{OR}\\
&\text{Condition 4.3: } (\chi>1/\eta) \quad \text{AND} \quad (\kappa\le -\eta\chi \ \text{ OR } \ \kappa\ge -\eta^2)
\end{aligned} \tag{17}
$$

Proof: It has already been assumed that $\kappa < 0$. The goal of this proof is to find the parameter range of κ so that the zeros in the zero loci of Fig. 2 do not lie on the RHS of the imaginary axis. It can be observed from these zero loci plots that there are three possible conditions (each with a corresponding value of κ) where the locus transitions from the LHS to the RHS, or vice versa, of the imaginary axis.

1. For a certain value of κ, the zero locus of $G_2(s)$ passes through the origin of the s-plane along the real axis (Figs. 2a, 2b, and 2c). In the process, the zero changes from real minimum phase (RMP) to RNMP, or vice versa. Mathematically, this value of κ can be found by substituting the Laplace variable $s = 0$ in Eq.(12).

$$\kappa A_2(s)+B_2(s)=0$$
$$\Rightarrow \kappa\left(s^2+2\zeta_v\omega_v s+\omega_v^{\,2}\right)_{s=0}+\left(s^2+2\zeta_u\omega_u s+\omega_u^{\,2}\right)_{s=0}=0$$
$$\Rightarrow \kappa\omega_v^{\,2}+\omega_u^{\,2}=0 \quad\Rightarrow\quad \kappa=-\eta^2$$

2. For a certain value of κ, the zero locus of $G_2(s)$ approaches negative infinity on the real axis of the s-plane and flips over to positive infinity and the zero changes from RMP to RNMP, or vice versa (Figs. 2a, 2b, and 2c). Mathematically, this corresponds to Eq.(12) having a single root, which happens when the coefficient of s^2 is zero.

$$\kappa A_2(s)+B_2(s)=0$$
$$\Rightarrow (\kappa+1)s^2+2\left(\kappa\zeta_v\omega_v+\zeta_u\omega_u\right)s+\kappa\omega_v^{\,2}+\omega_u^{\,2}=0$$
Setting coefficient of s^2 to zero
$$\Rightarrow (\kappa+1)=0 \quad\Rightarrow\quad \kappa=-1$$

3. For a certain value of κ, the zero locus of $G_2(s)$ crosses the imaginary axis at a non-zero frequency (Figs. 2a and 2c) and the zeros transition from complex minimum phase (CMP) to CNMP, or vice versa. Mathematically, this corresponds to purely imaginary roots of Eq.(12), which happens when the coefficient of s is zero.

$$\Rightarrow \left(\kappa\zeta_v\omega_v+\zeta_u\omega_u\right)=0 \quad\Rightarrow\quad \kappa=-\eta\chi$$

Referring to Fig. 2a, where $\chi < \eta$, it can been seen that the zero locus remains on the LHS real axis when $\kappa < -1$ and crosses back into the LHS of the imaginary axis when $\kappa \ge -\eta\chi$. In Fig. 2b, which corresponds to the parameter range $\eta \le \chi \le 1/\eta$, there are no CNMP zeros in the transfer function $G_2(s)$ (based on **Result 3**). The zero locus flips from negative infinity to positive infinity on the real axis when $\kappa = -1$ and returns to the LHS real axis, crossing the origin when $\kappa = -\eta^2$. Thus, no NMP zeros will occur if $\kappa \le -1$ or if $\kappa \ge -\eta^2$. Fig. 2c shows the zero locus of $G_2(s)$ when $\chi > 1/\eta$, and it can be seen that NMP zeros will not occur if $\kappa \le -\eta\chi$ or if $\kappa \ge -\eta^2$. These conditions form the necessary and sufficient conditions of Eq.(17). Based on **Result 4** and Figure 2, the following observations can be made about the NMP zero dynamics of the two-DoF damped flexible system:

1. Each condition listed in Eq.(17) is individually sufficient but not necessary. For example, Condition 4.1, by itself, is a sufficient condition for the elimination of all NMP zeros. However, Condition 4.1, by itself, is not necessary because even if this condition is not met, NMP zeros can still be eliminated via other non-overlapping conditions such as Condition 4.2 or Condition 4.3.
2. Each sufficient condition in Eq.(17) comprises of parameter ranges that are essential and broadest possible. For each of these conditions, one can write various inferior conditions with narrower parameter ranges that would also be sufficient conditions. For example, based on Condition 4.1,

one can say that [$\chi < \eta$ AND $\kappa < -2$] is also a sufficient condition for the elimination of NMP zeros.

3. As shown by the zero loci of Fig. 2, the entire range of the system parameters comprising of modal residues, frequencies, and damping ratios is covered in this analysis. Therefore, the sufficient conditions in Eq.(17) is a complete list of all possible sufficient conditions. In other words, there are no other sufficient conditions for which one can guarantee the elimination of NMP zeros. As a result, these three conditions when considered together, i.e., [Condition 4.1 OR Condition 4.2 OR Condition 4.3], form a necessary condition for the elimination of NMP zeros in a two-DoF damped flexible system when the modal residue signs are not same ($\kappa < 0$).

4. The mathematical form of the conditions in Eq.(17) is due to our choice of parameterization. The normalized parameters κ, χ, and η that are defined in terms of system parameters and used to provide the conditions in Eq.(17) could have been defined differently. For example, instead of considering κ (= α_u / α_v) as the varying parameter to plot the zero locus of $G_2(s)$, one could use a different normalized parameter defined by α_v / α_u. The zero locus could have been plotted as a function of ratio of modal frequencies or ratio of modal damping ratios. While the resulting mathematical form of Eq.(17) may be different in that case, the conditions would effectively be the same in terms system parameters. In other words, the conditions in Eq.(17) are unique.

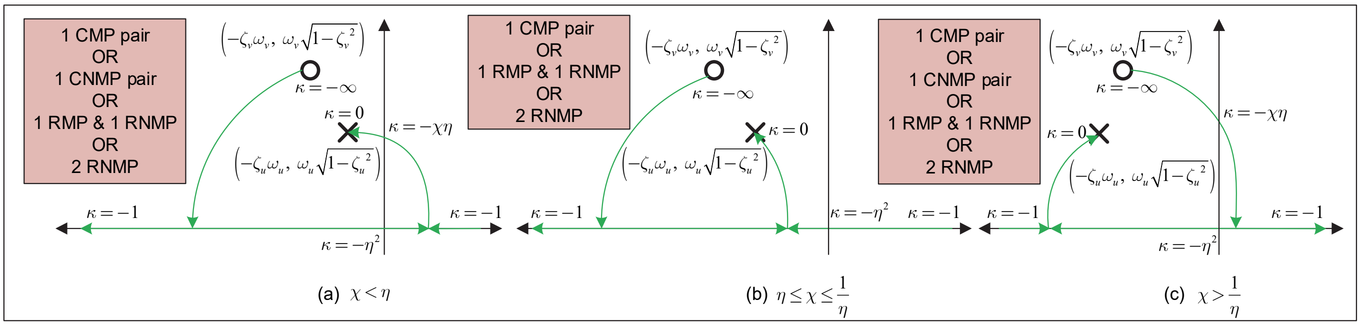

Fig. 2. Zero Loci of $G_2(s)$ for different ranges of χ as κ varies from $-\infty$ to 0. Zeros and poles of $T_2(s)$ provide the starting and ending locations, respectively, of these zero loci

5. CONLUSION AND FUTURE WORK

In this paper, a sufficient condition and necessary and sufficient conditions for the elimination of NMP zeros are presented for multi-DoF and two-DoF damped flexible LTI systems respectively. A sufficient condition for the elimination of non-minimum phase (NMP) zeros from the SISO transfer function of multi-DoF damped flexible LTI systems is that all the modal residue signs are same. It was shown in (Martin 1978) that multi-DoF undamped flexible LTI systems (with same modal residue signs) do not exhibit NMP zeros. Therefore, the addition of passive viscous damping does not lead to NMP zeros in multi-DoF damped flexible LTI systems (with same modal residue signs). Furthermore, it is shown that if all the poles of the multi-DoF damped flexible LTI systems (with same modal residue signs) are underdamped, the zeros will be necessarily complex minimum phase (CMP).

However, this is not the case when the modal residue signs are different. In the case of a two-DoF damped flexible LTI system, the addition of passive viscous damping can lead to the occurrence of complex non-minimum phase (CNMP) zeros. Therefore, a sufficient condition (**Result 3**), expressed in terms of the ratio of modal damping ratios (χ), and modal frequencies (η), is provided to guarantee the elimination of CNMP zeros for any value of modal residues. Furthermore, a complete list of all possible sufficient conditions to eliminate all NMP zeros is derived in **Result 4** using the zero loci of the two-DoF damped flexible LTI system which span the entire range of system parameters. The conditions in **Result 4** are shown to be individually sufficient, and together necessary, to guarantee the elimination of all NMP zeros.

The necessary and sufficient conditions (**Result 4**) derived in this paper can be used to design passive viscous damping strategies and select damping parameters for two-DoF damped flexible LTI systems. This would ensure improved dynamic performance by placing the poles on the LHS of the imaginary axis and eliminating NMP zeros from system dynamics simultaneously. There are two limitations to the modeling work presented in this paper. Firstly, the results and graphical insights are restricted to classically damped flexible systems. Secondly, the necessary and sufficient conditions for the elimination of NMP zeros have only been investigated for a two-DoF damped flexible LTI system. In the future, the zero dynamics of non-classically damped flexible systems will be investigated. The necessary and sufficient conditions for a three-DoF damped flexible LTI system will be derived. Experimental results will be provided based on these conditions to show how passive viscous damping can be applied to parameter varying undamped flexure mechanisms (Cui et al. 2017) to guarantee the elimination of NMP zeros over a wide range of system parameters.

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References

Awtar, S. and Parmar, G. (2013) Design of a Large Range XY Nanopositioning System, *Journal of Mechanisms and Robotics,* 5(2), p.021008.

Caughey, T. K. and O'Kelly, M. E. J. (1965) Classical normal modes in damped linear dynamic systems, *Journal of Applied Mechanics,* 32(3), 583-588.

Chalhoub, N. G. and Ulsoy, A. G. (1986) Dynamic Simulation of a Leadscrew Driven Flexible Robot Arm and Controller, *Journal of Dynamic Systems, Measurement, and Control,* 108(2), 119-126.

Chang, J.-Y. (2007) Hard disk drive seek-arrival vibration reduction with parametric damped flexible printed circuits, *Microsystem Technologies,* 13(8), 1103-1106.

Choi, K. B. and Lee, J. J. (2005) Passive compliant wafer stage for single-step nano-imprint lithography, *Review of scientific instruments,* 76(7), p.075106.

Cui, L., Okwudire, C. and Awtar, S. (2017) Modeling complex nonminimum phase zeros in flexure mechanisms, *Journal of Dynamic Systems, Measurement, and Control,* 139(10), p.101001.

Duffour, P. and Woodhouse, J. (2004) Instability of systems with a frictional point contact. Part 1: basic modelling, *Journal of Sound and Vibration,* 271(1-2), 365-390.

Engelen, K., Ramon, H., Saeys, W., Franssens, W. and Anthonis, J. (2007) Positioning and tuning of viscous damper on flexible structure, *Journal of Sound and Vibration,* 304(3), 845-862.

Feng, G., Fook Fah, Y. and Ying, Y. (2005) Modeling of hard disk drives for vibration analysis using a flexible multibody dynamics formulation, *IEEE Transactions on Magnetics,* 41(2), 744-749.

Freudenberg, J. and Looze, D. (1985) Right half plane poles and zeros and design tradeoffs in feedback systems, *IEEE Transactions on Automatic Control,* 30(6), 555-565.

Friedmann, P. P. and Millott, T. A. (1995) Vibration reduction in rotorcraft using active control - A comparison of various approaches, *Journal of Guidance, Control, and Dynamics,* 18(4), 664-673.

Giurgiutiu, V. (2000) Review of Smart-Materials Actuation Solutions for Aeroelastic and Vibration Control, *Journal of Intelligent Material Systems and Structures,* 11(7), 525-544.

Hastings, G. G. and Book, W. J. (1986) Reconstruction and robust reduced-order observation of flexible variables, in *ASME Winter Annual Meeting*, Annaheim, CA, 11-16.

Hoagg, J. B., Chandrasekar, J. and Bernstein, D. S. (2006) On the Zeros, Initial Undershoot, and Relative Degree of Collinear Lumped-Parameter Structures, *Journal of Dynamic Systems, Measurement, and Control,* 129(4), 493-502.

Hu, Q. (2008) Sliding mode maneuvering control and active vibration damping of three-axis stabilized flexible spacecraft with actuator dynamics, *Nonlinear Dynamics,* 52(3), 227-248.

Martin, G. D. (1978) *On the control of flexible mechanical systems,* Stanford University.

Meirovitch, L. (1967) *Analytical methods in Vibration,* New York: The Macmillan Company.

Pang, S. T., Tsao, T. and Bergman, L. A. (1993) Active and Passive Damping of Euler-Bernoulli Beams and Their Interactions, *Journal of Dynamic Systems, Measurement, and Control,* 115(3), 379-384.

Rath, S., Cui, L. and Awtar, S. (2021) On the Zeros of an Undamped Three-DOF Flexible System, *ASME Letters in Dynamic Systems and Control,* 1(4), p.041010.

Roesset, J. M., Whitman, R. V. and Dobry, R. (1973) Modal analysis for structures with foundation interaction, *Journal of the Structural Division,* 99(3), 399-416.

Thomas, A. E., Dickerson, S. L. and Wayne, B. J. (1986) On the Transfer Function Modeling of Flexible Structures with Distributed Damping, in *ASME Winter Annual Meeting*, 23-30.

Thompson, A. G. (1981) Optimum tuning and damping of a dynamic vibration absorber applied to a force excited and damped primary system, *Journal of Sound and Vibration,* 77(3), 403-415.

Thomson, W. T., Calkins, T. and Caravani, P. (1974) A numerical study of damping, *Earthquake Engineering and Structural Dynamics,* 3(1), 97-103.

Tsai, N.-C. (1974) Modal damping for soil-structure interaction, *Journal of the Engineering Mechanics Division,* 100(2), 323-341.

Tsai, T.-C., Tsau, J.-H. and Chen, C.-S. (2009) Vibration analysis of a beam with partially distributed internal viscous damping, *International Journal of Mechanical Sciences,* 51(11), 907-914.

Varanasi, K. K. and Nayfeh, S. A. (2004) The Dynamics of Lead-Screw Drives: Low-Order Modeling and Experiments, *Journal of Dynamic Systems, Measurement, and Control,* 126(2), 388-396.

Varanasi, K. K. and Nayfeh, S. A. (2006) Damping of flexural vibration using low-density, low-wave-speed media, *Journal of Sound and Vibration,* 292(1), 402-414.

Wang, J., Dong, X., Barry, O. R. and Okwudire, C. (2021) Friction-induced instability and vibration in a precision motion stage with a friction isolator, *Journal of Vibration and Control*, p.1077546321999510.