Extensible-Link Kinematic Model for Characterizing and Optimizing Compliant Mechanism Motion

We present an analytical model for characterizing the motion trajectory of an arbitrary planar compliant mechanism. Model development consists of identifying particular material points and their connecting vectorial lengths in a manner that represents the mechanism topology; whereby these lengths may extend over the course of actuation to account for the elastic deformation of the compliant mechanism. The motion trajectory is represented within the model as an analytical function in terms of these vectorial lengths, whereby its Taylor series expansion constitutes a parametric formulation composed of load-independent and load-dependent terms. This adds insight to the process for designing compliant mechanisms for high-accuracy motion applications because: (1) inspection of the load-independent terms enables determination of specific topology modifications that reduce or eliminate certain error components of the motion trajectory; and (2) the load-dependent terms reveal the polynomial orders of principally uncorrectable error components in the trajectory. The error components in the trajectory simply represent the deviation of the actual motion trajectory provided by the compliant mechanism compared to the ideally desired one. A generalized model framework is developed, and its utility demonstrated via the design of a compliant microgripper with straight-line parallel jaw motion. The model enables analytical determination of all geometric modifications for minimizing the error trajectory of the jaw, and prediction of the polynomial order of the uncorrectable trajectory components. The jaw trajectory is then optimized using iterative finite element simulations until the polynomial order of the uncorrectable trajectory component becomes apparent; this reduces the error in the jaw trajectory by 2 orders of magnitude over the prescribed jaw stroke. This model serves to streamline the design process by identifying the load-dependent sources of trajectory error in a compliant mechanism, and thereby the limits with which this error may be redressed by topology modification. [DOI: 10.1115/1.4026269]

Keywords: compliant mechanism, accuracy, path generation, flexure, microgripper, topology optimization, extensible-link, kinematic model, load dependency

1 Introduction

By virtue of having zero backlash and no Coulomb friction, compliant mechanisms are particularly suited for executing precision tasks requiring high-accuracy motion, including micro- and nano-manipulation [1–4], mechanical transduction [5,6], component alignment [7–9], and metrology [10,11]. Moreover, compliant mechanisms are ubiquitous in microfabricated devices and systems because monolithic construction is easily achieved by lithography and etching of silicon wafers [12].

A compliant mechanism is typically designed to provide a desired motion trajectory within a set of constraints, which may include the available mechanism area (i.e., the “footprint”), the means of actuation, the material properties, and the capabilities of the fabrication process. As shown schematically in Fig. 1, a compliant mechanism must be designed to fit within the mechanism area, be anchored at the available ground location(s), and comprise the “trajectory body,” with respect to which the desired motion trajectory, \( \Omega(d) \), is defined. This trajectory, \( \Omega(d) \), is the translation of a particular material point on, or the translation/rotation of, the trajectory body; and is traced out by actuation of the compliant mechanism via an applied load. This load may be applied by various means, including: displacement of a linear/rotary actuator; induced strain of a portion of the compliant mechanism (thermal, piezoelectric, etc.); or by an applied electrostatic and/or electro-magnetic body force (comb-drive actuator, voice-coil actuator, etc.). The desired trajectory can be expressed as a function of a stroke parameter, \( \xi \), which may represent either the applied load or the motion component of a body in the compliant mechanism that is critical to defining the desired trajectory. Here, \( \xi_0 \) corresponds to the undeformed state of the compliant mechanism. The design task for high-accuracy motion applications represented by Fig. 1 is therefore to design a compliant mechanism that achieves the desired trajectory, \( \Omega_d(\xi) \), given a prescribed mechanism area (dashed rectangle), ground location, and load application.

An important part of this design process is to evaluate a candidate compliant mechanism by determining the accuracy with which it can trace the desired trajectory. This evaluation is important because the extent to which the desired motion trajectory can be realized via the compliant mechanism determines its efficacy for the motion application. Generally speaking, this motion trajectory is dependent on the load application, as well as the
mechanism topology and shape. These dependencies are coupled, which adds difficulty to understanding and predicting the exact motion characteristics of the mechanism.

Topology symmetry is often utilized to avoid this complication, yet this is only feasible where there is adequate available mechanism area, and only for certain desired trajectories (i.e., straight lines). Application-specific requirements often place strict limitations on some aspects of the design domain, such as: mechanism size; location and orientation of the applied load; type of actuator for applying the load; available ground locations; location of, and attachment points to, the trajectory body; and material selection. If topology symmetry is not feasible given the shape of the desired attachment points to, the trajectory body; and material selection.

Several analytical methods have been developed that address this evaluation task. For instance, the Pseudo-Rigid-Body Model (PRBM) [13,14] expedites the synthesis and design iteration of a candidate compliant mechanism topology by means of an analogous rigid-body kinematic linkage. While there are specific combinations of mechanism topologies and loading conditions for which PRBM provides an accurate trajectory approximation, the exact trajectory of a compliant mechanism need not be entirely representable by the motion of a rigid-body kinematic linkage. This is because a fixed topology rigid-body kinematic model does not adequately capture all the elastic deformations that arise over the desired range of motion of the mechanism. As an alternative to PRBM, closed-form analytical solutions have been developed for compliant mechanisms built from beam flexures to capture kinematic, elastic, and elastokinematic effects [15,16]. But this modeling effort addresses a limited range of beam shapes, and extending it to any general mechanism topology and beam shapes remains a challenge [17].

While these two analytical methods are certainly useful, the nature of the applied load, and/or the intricacy of mechanism topology/shape may not allow these methods to accurately evaluate trajectory accuracy. For such cases, finite elements (FE) modeling is the only well-established tool for quantitative evaluation of the exact trajectory accuracy of a candidate compliant mechanism. FE modeling is therefore often utilized in conjunction with the aforementioned analytical models, as well as with recursive numerical procedures that integrate one or more the following design steps: (1) synthesis of a candidate compliant mechanism topology; (2) evaluation of the mechanism’s trajectory accuracy; and (3) optimization by means of modifying the mechanism topology and shape so as to minimize trajectory inaccuracy. Approaches include multi-criteria [18,19], continuous material distribution [20,21], and genetic [22,23] numerical optimization algorithms. With sufficient FE simulation iterations, it is possible, in many cases, to modify a candidate compliant mechanism’s topology and shape so that it exhibits sufficient trajectory accuracy. However, a residual error trajectory often exists after optimization, in terms of the mechanism’s ability to trace the desired motion trajectory.

Importantly, neither FE modeling nor the recursive optimization methods elicit an intuitive understanding regarding the existence, magnitude, or characteristic form of this residual error trajectory. The designer is therefore left without a clear understanding regarding why this residual error exists, or to what extent it may, in principle, be redressed. It is possible for considerable time to be spent modifying the mechanism’s topology and shape in an attempt to redress this residual error trajectory, which may in fact be fundamentally uncorrectable due to some aspect of the mechanism topology. If unsuccessful in sufficiently reducing the error trajectory, and being no more informed as to its source within the compliant mechanism, the designer is left to simply “guess” either: (1) a new model if synthesizing by PRBM, or (2) new/additional initial conditions if synthesizing by a recursive numerical method.

To address this difficulty, we have developed an analytical approach that aids in understanding and evaluating the motion trajectory characteristics of an arbitrary planar compliant mechanism designed to accomplish a high-accuracy motion task (Fig. 1). Here, a model is created that consists of vectorial lengths spanning between selected material points which represent locations of connection between segments comprising the compliant mechanism. This may be intuitively visualized as a kinematic linkage, wherein the link lengths extend over the course of mechanism actuation to account for the elastic deformation of the compliant mechanism under the actuation loading. Within this model framework, the trajectory of the compliant mechanism, as well as extensions of the link lengths, are expressed as analytical functions with respect to a stroke parameter, $\zeta$. A Taylor series expansion of the mechanism’s trajectory is then performed with respect to $\zeta$. This enables the trajectory to be represented by two parametrically separated motion components: rigid-body terms that contain only link lengths and orientations related to the undeformed state of the compliant mechanism; and deformation terms that, in addition, contain link extension components.

The significance of this parametric representation is that the rigid-body terms and the deformation terms comprise load-independent and load-dependent components of the compliant mechanism’s motion trajectory, respectively. Because the rigid-body terms are both load-independent and solely described by the undeformed state of the compliant mechanism topology, they constitute a well-defined motion trajectory component that is entirely specifiable by design. Conversely, the deformation terms capture all load-geometry interdependencies, which necessarily arise over the course of mechanism actuation. Therefore, their magnitudes are also dependent on the mechanism’s shape. Within this framework, trajectory optimization may be regarded as a procedure in which the summation of the motion contributions from the rigid-body terms and deformation terms is designed to produce the desired trajectory.

This approach can streamline the compliant mechanism design process because: (1) inspection of the rigid-body terms enables specific topology modifications to be determined for minimizing the error trajectory; and (2) the polynomial orders of principally uncorrectable trajectory components are captured by the deformation terms. While quantitative optimization of the compliant mechanism trajectory must still be performed by iterative FE simulation, all geometric correction parameters for the mechanism topology, as well as the characteristic form of the residual error trajectory, are known beforehand. As a result, some ineffective mechanism designs and topology modifications may be disregarded without FE simulation, and time is not spent attempting to redress trajectory errors that are principally uncorrectable via topology modification. This serves to reduce the amount of time and number of numerical iterations necessary to arrive at a compliant mechanism that meets or exceeds the requirements for motion accuracy.

This paper presents the analytical framework for an extensible-link kinematic model (ELKM) and details how it may be utilized, in conjunction with FE modeling, as a design and trajectory optimization method (Sec. 2). Its utility is then demonstrated in a case study (Sec. 3), where a compliant gripping mechanism with a
straight-line parallel jaw trajectory is designed. The model is used to determine the polynomial order of the jaw’s residual error trajectory, and to guide the process of optimizing the jaw motion by iterative FE simulation. The model is then summarized and discussed in context of the case study results (Sec. 4).

2 Generalized Model

2.1 Definitions and Concepts for Model Development. The desired motion trajectory, \( \Omega_d(\xi) \) (denoted in a global coordinate frame), of a candidate compliant mechanism, is considered a function of a stroke parameter, \( \xi \) (Fig. 1). The actual motion trajectory of this corresponding material point/body on the compliant mechanism is defined by \( \Omega_c(\xi) \), which is a function of the same stroke parameter, \( \xi \). Ideally, the compliant mechanism trajectory, \( \Omega_c(\xi) \), and the desired trajectory, \( \Omega_d(\xi) \), are equivalent. Therefore, the error trajectory, \( \delta(\xi) \), is the difference \( \Omega_c(\xi) - \Omega_d(\xi) \). Here, \( \delta_0 \) corresponds to the undeformed state of the compliant mechanism; and the deformed states within the range of actuation are defined by \( \xi - \xi_0 \). Because this actuation range is limited by finite material strain, it is likely that \( \Omega_d(\xi) \) and \( \Omega_c(\xi) \) may each be entirely represented by a single smooth continuous function. However, in general, \( \Omega_d(\xi) \) and \( \Omega_c(\xi) \) may each comprise a set of piece-wise smooth continuous functions defined with respect to global coordinates, whereby the following analysis would be performed for each function in the set.

Generally speaking, the distance, \( l \), between any two material points in a deformable continuum body (Fig. 2(a)) may admit decomposition into (Eq. 1(a)): an initial length, \( l_0 \), corresponding to the undeformed state, \( \xi_0 \), and an extensible component, \( f(\xi) \), expressed as a function of \( \xi \). The extensible component simply describes the change in distance between the two material points over the course of deformation. This imparts a requirement that \( f(\xi) \) be a continuous function that is equal to zero at the undeformed state, \( \xi_0 \) (Eq. 1(b)). Regardless of the actuation of a compliant mechanism, \( f(\xi) \) is attributable to the load-dependent elastic and geometric deformation that arises during loading.

\[
\begin{align*}
\quad l & = l_0 + f(\xi) \quad (1a) \\
\quad f(\xi_0) & = 0 \quad (1b)
\end{align*}
\]

Significance, from a design perspective, is gained by selecting particular material points that represent locations of connection between segments comprising a compliant mechanism topology—namely: (1) the centers of thin compliant hinges between significantly wider sections (i.e., lumped-compliance shapes) (Fig. 2(b)); and (2) the end-points of compliant beams (i.e., distributed-compliance shapes) (Fig. 2(c)). Notice that no restriction has been placed on the deformation or shape complexity of these segments (Figs. 2(b) and 2(c)) for the decomposition of Eq. (1) to be applicable. It is in this context an “arbitrary” compliant mechanism is defined as a composition of such lumped- and distributed-compliance segments (Figs. 2(b) and 2(c)), whose configuration (i.e., location and orientation of \( l_0 \)) represents the mechanism topology. The particular shape of the compliant segments affects only the extensible components, \( f(\xi) \).

The complete set of selected material points for the model therefore consists of: (1) the locations of connection between compliant segments (Figs. 2(b) and 2(c)); and (2) the point defining the motion trajectory, \( \Omega_c(\xi) \) (Fig. 1). The model simply comprises the connected vectorial lengths, \( l \), which represent the material spanning between the selected material points. This is illustrated in Fig. 3 for the lumped-compliance (Fig. 3(a)) and distributed-compliance (Fig. 3(b)) shapes of an example compliant mechanism topology (Fig. 3(c)). Notice that the material of the trajectory body spans distances \( e \) and \( f \) as well. Because both shapes (Figs. 3(a) and 3(b)) have identical coordinate locations where the segments comprising each compliant mechanism are connected (i.e., identical topologies), both share identical model arrangements at the undeformed state, \( \xi_0 \) (i.e., the same \( l_0 \) lengths in Fig. 3(c)). The extensible components, \( f(\xi) \), of the distances, \( l \), comprising the models for each respective shape, however, may change differently across the range actuation, \( \xi - \xi_0 \).

The motions of the vectorial lengths, \( l \), may be equivalently envisioned as an analogous kinematic mechanism (Fig. 3(d)), wherein extensible links are connected by pin joints residing at the selected material points (i.e., an extensible-link kinematic
the latter is henceforth adopted because it provides an intuitive context—especially for visualizing the load-independent motion component in the following derivation. It also reinforces the applicability of classical kinematic analysis in performing the vector algebra to construct the model trajectory, \( \Omega_i(\zeta) \) in terms of the lengths and orientations of the extensible links, and as a function of the same stroke parameter, \( \zeta \) (Fig. 3(d)). The trajectories \( \Omega_i(\zeta) \) and \( \Omega_i(\zeta) \) are therefore equivalent so long as the links in the model extend in the proper manner over the range of actuation, \( \zeta_{0} - \zeta_{0} \).

While determining explicit formulas for the extensible components is generally nontrivial, the initial link lengths (i.e., \( l_0 \)) are identified simply from the undeformed state of the compliant mechanism, \( \zeta_{0} \), according to this straightforward procedure. It will be shown that explicit formulation of the extensible components is inconsequential for distinguishing the load-independent and load-dependent motion contributions to the compliant mechanism trajectory. It is for this reason that the following derivation considers extensible length components, \( f(\zeta) \), simply as smooth continuous functions, thereby granting admission to treat \( \Omega_i(\zeta) \) as equal to \( \Omega_i(\zeta) \). The design goal, again, is to have these equivalent trajectories, \( \Omega_i(\zeta) \) and \( \Omega_i(\zeta) \), match the desired trajectory, \( \Omega_i(\zeta) \).

2.2 Analytical Formulation. Without admitting any link length decomposition, and without loss of generality, \( \Omega_i(\zeta) \) may be expressed by its Taylor series expansion (Eq. (2)) with respect to \( \zeta \), about the undeformed state of the mechanism (\( \zeta_{0} \)). All coefficients in the expansion, \( k_n \), are functions of the extensible link lengths and orientations. Because the range of actuation, \( \zeta_{0} - \zeta_{0} \), is finite, it can be expected that the expansion may be truncated as an \( n \)-th order polynomial, where higher-order terms are negligible.

\[
\Omega_i(\zeta) = \sum_{n=0}^{\infty} \Omega_{n}(\zeta) = \sum_{n=0}^{\infty} \frac{(\zeta - \zeta_{0})^n}{n!} \Omega_i(\zeta)
\]

By now admitting the decomposition of Eq. (1), the \( k_n \) coefficients in Eq. (2) become functions of the initial lengths, \( l_n \), as well as \( \zeta \) due to the extensible components (Eq. (3)). Each \( k_n \) coefficient may therefore be represented by its respective Taylor series expansion, with respect to \( \zeta \), about the undeformed state, \( \zeta_{0} \). Based on the magnitude of the extensible components, \( f(\zeta) \), over the stroke, the Taylor series expansion for each \( k_n \) coefficient may be truncated at some \( j \)-th order, with respect to \( \zeta_{0} - \zeta_{0} \), such that the higher-order terms, \( k_{n, m, j} \), are negligible.

\[
\begin{align*}
\Omega_i(\zeta) & \approx \sum_{n=0}^{j} \sum_{m=0}^{j \cdot n} k_{n, m}(\zeta - \zeta_{0})^m \Omega_i(\zeta) \nonumber \\
& \approx \sum_{j=0}^{j} \sum_{j=0}^{j \cdot n} k_{j, n}(\zeta - \zeta_{0})^j \Omega_i(\zeta) \nonumber \\
& = \Omega_i(\zeta) \nonumber \\
Omega_i(\zeta) & = \sum_{j=0}^{j} \sum_{j=0}^{j \cdot n} k_{j, n}(\zeta - \zeta_{0})^j \Omega_i(\zeta)
\end{align*}
\]

As a result of Eq. 1(b), the first term in each \( k_n \) Taylor series, \( k_{n, 0} \), contains only \( l_0 \) lengths; these are referred to as rigid-body terms. Their constitution is unaffected by the extensible components, \( f(\zeta) \), and therefore they collectively represent a motion trajectory component that is independent of the deformation of the compliant segments (i.e., depends only on their configuration). Furthermore, these \( k_{n, 0} \) terms exactly constitute the rigid-body motion trajectory, in series-representation, of the kinematic mechanism comprising only \( l_0 \) link lengths (i.e., as if all \( f(\zeta)\equiv 0 \); and this is denoted by \( \Omega_{R,B}(\zeta) \) in Eq. (3). The order of truncation, \( i \), is therefore determined by the number of significant terms in \( \Omega_{R,B}(\zeta) \) for the range of actuation, \( \zeta_{0} - \zeta_{0} \). Thus, \( \Omega_{R,B}(\zeta) \) represents a component of the compliant mechanism trajectory, \( \Omega_i(\zeta) \), that is independent of the applied load, segment shapes, and material properties; and it may be derived simply from the compliant mechanism’s undeformed state, \( \zeta_{0} \), using rigid-body kinematics.

All remaining terms in each \( k_n \) Taylor series, \( k_{n, m, j} \) (Eq. (3)), contain initial link lengths, \( l_n \), as well as derivatives of the extensible components up to the \( j \)-th order, evaluated at \( \zeta_{0} \) (i.e., \( f(\zeta_{0})^{(m)} \)). These are referred to as deformation terms. Their constitution is of a form such that they require the existence of extensible components, \( f(\zeta) \), to be nonzero valued; and their magnitudes correspond to the magnitudes of the extensible component derivatives (i.e., \( f(\zeta_{0})^{(m)} \)). This is shown explicitly in the case study (Eq. (12), Sec. 3). The motion trajectory of a compliant mechanism, \( \Omega_i(\zeta) \), therefore contains a load-dependent component that is represented within the model by the collective contribution of the deformation terms, \( \Omega_{D}(\zeta) \) (Eq. (3)). Hence, the values of the deformation terms: (1) capture the load-deformation interdependencies arising over the course of mechanism actuation; and (2) reflect the shape of the compliant mechanism.

The entire series expansion is now written explicitly, grouping like-ordered \( k_{n} \) terms (Eq. (4)). For clarity, these terms are expressed as functions of \( \Delta \zeta \), which represents the displacement from the mechanism’s undeformed state. The rigid-body, \( k_{n, 0} \), and deformation, \( k_{n, m, j} \) (notated in bold), terms are now represented in a parametric form showing that there is one rigid-body term per polynomial order, terminating at the \( j \)-th order with respect to \( \Delta \zeta \). Deformation terms may range from 1st order to \( j \)-th order, for some \( j > i \).

\[
\begin{align*}
\Omega_i(\Delta \zeta) & \approx \sum_{n=0}^{j} \sum_{m=0}^{j \cdot n} k_{n, m}(\zeta - \zeta_{0})^m \Omega_i(\zeta) \nonumber \\
& \approx \sum_{j=0}^{j} \sum_{j=0}^{j \cdot n} k_{j, n}(\zeta - \zeta_{0})^j \Omega_i(\zeta) \nonumber \\
& = \sum_{j=0}^{j} \sum_{j=0}^{j \cdot n} k_{j, n}(\zeta - \zeta_{0})^j \Omega_i(\zeta)
\end{align*}
\]

Similarly, the desired motion trajectory, \( \Omega_{D}(\zeta) \), may be represented by its Taylor series expansion, with respect to \( \zeta \), about the undeformed state of the mechanism, \( \zeta_{0} \) (Eq. (5)).

\[
\Omega_{D}(\Delta \zeta) = \sum_{n=0}^{j} d_{n}(\Delta \zeta)^n = d_0 + d_1 \Delta \zeta + d_2 \Delta \zeta^2 + \ldots
\]

The summation of rigid-body and deformation terms per polynomial order (Eq. (4)) is ideally equivalent to that of the desired trajectory (Eq. (5)). Since the actual compliant mechanism motion trajectory, \( \Omega_i(\zeta) \), and the model trajectory, \( \Omega_i(\zeta) \), may be considered equivalent (i.e., \( \Omega_i(\zeta) = \Omega_i(\zeta) \)), the error trajectory, \( \Omega_i(\zeta) - \Omega_i(\zeta) \), may also be written as the difference \( \Omega_i(\zeta) - \Omega_i(\zeta) \) (Eq. (6)).
Fig. 4 Compliant mechanism design procedure utilizing the ELKM

Note that, as the magnitudes of the extensible components approach zero (i.e., \( f(\xi) \to 0 \)), the magnitudes of the deformation terms, \( k_{n,0} \cdot f(\xi) \), also approach zero (i.e., \( \lim_{f(\xi) \to 0} k_{n,0} \cdot f(\xi) = 0 \)). Here, the model’s motion trajectory, \( \Omega_{C}(\xi) \), approaches the rigid-body kinematic trajectory, \( \Omega_{RB}(\xi) \). This limiting case corresponds to a limiting compliant mechanism shape: extreme lumped compliance, such that both the length and bending stiffness (i.e., thickness) of the compliant hinges approach zero. At this limit, the motion trajectory of the compliant mechanism, \( \Omega_{C}(\xi) \), becomes equivalent to the motion trajectory of the rigid-body terms, \( \Omega_{RB}(\xi) \).

An important observation here is that the deformation terms for any physical compliant mechanism will be nonzero; their magnitudes may only be minimized by lumped-compliance shapes or maximized by distributed-compliance shapes. And, these limits of compliance distribution are also typically bounded by practical considerations such as: (1) the material yield strain, given the range of mechanism actuation, \( \xi \); (2) the available mechanism area (Fig. 1); and (3) the details of the chosen fabrication process. In contrast, the magnitudes and signs of the rigid-body terms are entirely determined by the locations of connection (i.e., kinematic constraint) between the segments comprising the compliant mechanism in its undeformed state, \( \xi \); and these locations can be altered within the available mechanism area (Fig. 1) by design. Hence from a design perspective, the values of the rigid-body terms may be considered “completely specifiable by design.”

By this insight, the procedure for minimizing the trajectory error (Eq. (6)) of a compliant mechanism—via modification of its topology—may be regarded as follows within the model framework: per polynomial order, the value of each rigid-body term is designed to compensate for the corresponding deformation terms so as to provide the correct overall motion trajectory of that order. In other words, the configuration of compliant segments (i.e., the mechanism topology) is designed to compensate for load-dependent trajectory components that arise during actuation. Within this context, Eq. (6) illustrates that the error trajectory only up to the \( i \)th order may be, in total or in part, redressed by modification of the mechanism topology. All trajectory components greater than or equal to the \((i+1)\)th order are exclusively deformation terms, and are therefore, in principle, uncorrectable (i.e., only minimizable). By the nature of Taylor series expansions, it is likely that the \((i+1)\)th order contribution will dominate that of any higher-order deformation terms over the finite actuation range, \( \xi > \xi_{0} \).

Given a specific candidate compliant mechanism, the effectiveness of redressing the error trajectory may be determined by inspection of the rigid-body terms within the model. The formulation for each \( k_{n,0} \) term indicates how its value may be modified by changing the initial lengths/orientations (i.e., \( l_{0} \) in Eq. (1a)). This thereby indicates how corresponding geometric changes to the compliant mechanism topology affect its trajectory, provided its shape remains largely invariant. Ideally, there would be \( i \)-number of unique geometric parameters available, each of which could be independently changed to modify the value of corresponding \( k_{n,0} \) terms. In this case, the residual trajectory error would consist of significant \((i+1)\)th to \( j \)th order polynomial components. Having less than \( i \)-number of suitable geometric modifications forces a tradeoff in the optimization procedure between the values of two or more \( k_{n,0} \) terms in order to minimize the error trajectory.

Recall that increasing the distribution of compliance increases the magnitude of the deformation terms, \( k_{n,0} \cdot f(\xi) \), without affecting the value of the rigid-body terms, \( k_{n,0} \cdot 0 \). Regarding trajectory optimization, this indicates that a larger distribution in compliance may amount to more extensive geometric modifications and a larger magnitude residual error trajectory. Given material strain limitations, this implies a tradeoff between range of motion (i.e., increased by larger compliance distribution) and trajectory accuracy.

Overall, the extensible-link kinematic model illustrates a departure from classical rigid-link mechanism design, in which a desired motion trajectory is, in principle, traceable without error if fabrication is perfect. In contrast, Eq. (6) shows that the material deformation of a compliant mechanism during actuation can affect the load-dependent error trajectory, even if the topology and shape could be fabricated with perfect accuracy. Generally speaking, the model shows that, although material deformation in a compliant mechanism inherently provides motion repeatability, it is a source of trajectory error, which is critical in high-accuracy motion applications (Fig. 1).

The model is useful for understanding the qualitative motion characteristics of compliant mechanisms and for guiding the design process. Notably, developing this model framework required no assumptions or constraints regarding the compliant mechanism shape, material properties, or load application. These specifications are all contained within the extensible components, \( f(\xi) \), which, again, are generally nontrivial to express analytically. The practical utility of the model therefore comes from not having to specify explicit equations for the extensible components in order to identify load-independent and load-dependent motion components; and this makes it amenable to analysis of compliant mechanisms that cannot otherwise be represented in an analytical closed-form. For such cases, FE simulation can be used to infer the values of extensible components, \( f(\xi) \), and deformation terms, \( k_{n,0} \cdot f(\xi) \). This thereby enables optimization of the trajectory error by iterative modification of the geometric parameters identified within the model. This procedure is summarized below, and depicted in Fig. 4. Note that it may be analytical until the last step, where iterative numerical simulation is used to minimize the error trajectory (Fig. 4, dashed box).

1. The formulations of the rigid-body terms, which constitute the trajectory \( \Omega_{RB}(\xi) \), are determined by kinematic
objects such as cells [25], gels [26], and assemblies of micro and mechanical tension/compression tests [24], and for gripping soft jaw trajectory. Straight-line jaw motion may be desired for micro-
rotation in this manner greatly simplifies the following analysis of the accuracy limits of gripper jaw trajectory.

It is important to keep in mind that the majority of the design process presented here is analytical. FE simulation is not unnecessary until the final design step, where it is utilized to iteratively modify geometric parameters identified within the compliant mechanism to minimize the error trajectory of the jaws (Fig. 10). These modifications were performed manually in the same CAD software program used to create the initial 2D compliant gripper contour. The CAD model was exported into ANSYS and meshed appropriately. Only 10s of nonlinear FE simulation iterations were necessary, meaning that the total time requirement for optimization amounted to less than half a day. The efficiency of this process is attributable to two main factors: (1) the appropriate geometric modifications as well as the polynomial order of the residual error trajectory were analytically determined before FE simulation; and (2) as a result, it was not necessary to implement an automated algorithmic optimization procedure, which may have taken more time and would not provide the same insights as to how the compliant mechanism topology affects the jaw error trajectory.

3.2 Parallelogram Mechanism

The trajectory of P1, for the parallelogram topology is modeled by the pivoting of a beam with extensible length \( l \) (Fig. 6, Eq. (8)).

\[
P_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l \sin \theta \\ l \cos \theta \end{bmatrix} \tag{8}
\]

It can be shown by kinematic analysis that the 2nd-order Taylor series expansion (i.e., \( i = 2 \) in Eq. (3)) of the rigid-body \( P_1 \) trajectory, about the nominal position, \( \theta_0 \), is sufficient to capture all significant motion contributions for small angular perturbations, \( \Delta \theta \) (Eq. (7)). This may be written as a 2nd-order polynomial function, \( y(\Delta x) \), in the global X-Y coordinate system (Eqs. (9) and (10)). Here, the \( y \)-direction motion is represented as a function of the gripping direction displacement, \( \Delta x \) (i.e., the stroke parameter), whereby the desired trajectory is: \( y(\Delta x) = \text{constant} \) (i.e., horizontal translation of \( P_1 \)).

\[
y(\Delta x) = l \sqrt{1 - \left( \frac{\sin \theta_0 + \Delta x}{l} \right)^2} \approx k_0 + k_1 \Delta x + k_2 \Delta x^2 \tag{9}
\]

\[
k_0 = \sqrt{l^2 - (l_0 \sin \theta_0)^2} \tag{10a}
\]

\[
k_1 = \frac{-l_0 \sin \theta_0}{\sqrt{l^2 - (l_0 \sin \theta_0)^2}} \tag{10b}
\]

\[
k_2 = \frac{-l^2}{2 \left( l^2 - (l_0 \sin \theta_0)^2 \right)^{3/2}} \tag{10c}
\]

3.3 Hoekens-Derived Mechanism

Now consider replacing beam \( l \) (Fig. 6) with a four-bar mechanism (Fig. 7). Here, the extensible-link kinematic model within the mechanism area consists of a closed four-bar mechanism (hatched shading) that defines the path of \( P_1 \), and a parallelogram-based mechanism that serves to replicate the motion of \( P_1 \) at \( P_2 \) (Fig. 7(a)). Again, the trajectory of \( P_1 \) is evaluated, which represents the translation of the jaw link. Referencing the classical Hoekens linkage as a starting point, the geometric parameters of the four-bar are defined as follows (Fig. 7(b)): \( a = \text{crank}, \quad g = \text{ground}, \quad b = \text{follower}, \quad h = \text{output}, \quad d = \text{extension of h to P}_1 \text{ at angle } \varphi \). The vector trajectory of \( P_1 \) is written in the global X-Y coordinate system (Eq. (13)) such that it consists of the input crank angle, \( \theta \), defined with respect to the X-axis as shown, and the kinematic linkage arrangement (i.e., extensible link lengths and angle \( \psi \)). The expression for the internal angle, \( \varphi \), is derived based on the kinematic constraint that the
ends of link \( h \) must coincide with the respective ends of links a and b.

\[
P_1 = \begin{bmatrix} x \\ y \end{bmatrix} = a \cos \theta \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix} + h \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix} + d \begin{bmatrix} \cos(\varphi + \psi) \\ \sin(\varphi + \psi) \end{bmatrix}
\]

\[
\varphi = \arcsin \left( \frac{b^2 - a^2 - h^2 - g^2 + 2ag \cos \theta}{\sqrt{(2ah \sin \theta)^2 + (2h(a \cos \theta - g))^2}} \right) - \arctan \left( \frac{a \cos \theta - g}{a \sin \theta} \right)
\]

By kinematic analysis, the \( P_1 \) trajectory may expressed as the function, \( y(\Delta x) \), in the global X-Y coordinate system, where its 2nd-order Taylor series expansion (i.e., \( i = 2 \) in Eq. (3)) is sufficient to capture all significant motion contributions for small angular perturbations, \( \Delta \theta \) (Eq. (7)). The series representation of \( y(\Delta x) \) here is therefore identical in form to Eq. (11), but with the following equations for the rigid-body, \( k_{1,0} \), terms (Eq. (14)). The equations for the deformation terms, \( k_{\text{non},0} \), are not shown here because they are lengthy and their computation is not necessary within the context of the following trajectory optimization procedure.

\[
k_{2,0} = - \frac{\left( R_2^2 h_0 (a_0 - g_0) + R_0 g_0 (a_0 + g_0) - h_0 a_0 \right)}{2h_0 d_0} h_0 + d_0 \cos \psi \frac{\left( R_0 h_0 (a_0 - g_0) + g_0 (a_0 + g_0) \right)}{2h_0 d_0} d_0 \sin \psi
\]

For brevity, these expressions (Eq. (14)) include the substitution \( \theta_0 = 180 \text{ deg} \), which was determined, by inspection, to be requisite for a symmetric \( P_1 \) trajectory about the undeformed state of the mechanism. Here, symmetric trajectories occur for the following kinematic relationships: \( \psi = 0^\circ \), \( h_0 = b_0 = d_0 \), considering equal angular perturbations of the crank (i.e., link a) about \( \theta_0 \). Additionally, perfect horizontal straight-line motion (i.e., \( k_{1,0} = k_{2,0} = 0 \)) is achieved for the following kinematic relationship: \( h_0 = b_0 = d_0 = 0 \), such that \( d_0 / a_0 = 4 \) and \( g_0 = b_0 - a_0 \). These link length relationships are slightly different than that of the classical Hoekens mechanism (i.e., \( l_0 / a_0 = 2.5 \)). To achieve the desired jaw range (0–400 mm) given the small-angle restriction (Eq. (7)), the link length \( a_0 = 1.67 \text{ mm} \) is required, which is practically feasible.

Analysis of the rigid-body terms (Eq. (14)) reveals that both the signed and magnitude of \( k_{1,0} \) and \( k_{2,0} \) may be controlled by independently changing \( \psi \) and \( g_0 \), respectively (Fig. 8). Therefore, the 1st- and 2nd-order trajectory errors caused by the corresponding-order deformation terms may be completely corrected by proper modification of these two geometric parameters. Hence, the residual error trajectory of the gripper jaw is predicted to be a 3rd-order polynomial that is, in principle, not correctable. Note that, even though the 3rd-order rigid-body term (i.e., \( k_{3,0} \)) is negligible, the cumulative contribution from the 3rd-order deformation terms (i.e., \( k_{3,1} + k_{3,2} + k_{3,3} \)) may be significant in magnitude.

So far, only the trajectory of \( P_1 \) has been considered. To achieve the desired gripper jaw motion, the \( P_1 \) trajectory must be duplicated at \( P_2 \) by the parallelogram linkage portion of the mechanism (Fig. 7(a)). The above analysis may be performed for this parallelogram linkage as well in order to express the motion of \( P_2 \) in a parametric form. But, for brevity and clarity, it is simply noted that all significant contributions to the trajectory of \( P_2 \) may be captured in the form on Eq. (11) because all links in the parallelogram linkage are subject to the small angle constraint (Eq. (7)). Therefore, by analogy, adjusting \( \psi \) (Fig. 7(a)) modifies the linear trajectory of \( P_2 \). This angle will be optimized in order to minimize gripper jaw rotation, \( \Delta \omega \); and as the FE results will show, this linear correction of the \( P_2 \) trajectory is sufficient.

The Hoekens-based kinematic mechanism (Fig. 7(a)) is translated into a lumped-compliance flexure mechanism (Fig. 9) now that all suitable geometric modifications have been determined. Here, the geometric centers of thin compliant hinges (Fig. 9) coincide with the locations of the pin joints in the kinematic model (Fig. 7(a)). The hinges have a cycloidal profile [33] which, compared to other compliant hinge contours, maximizes in-plane rotational compliance and translational stiffness for a prescribed angular deflection limit (Eq. (7)) and allowable material strain. For microfabrication of the gripper from a silicon wafer, a strain limit of 0.5% [34] is chosen. Note that a lumped-compliance shape has been chosen in order to minimize the magnitude of the deformation terms, \( k_{\text{non},0} \), and thereby the magnitude of the residual error trajectory.

The jaw trajectory of the compliant mechanism is now optimized by iterative manual adjustment of the geometric parameters (i.e., \( \Delta \psi, \Delta g_0, \Delta \phi \)) and evaluated using nonlinear FE simulation in ANSYS. The rotation (Fig. 10(a)) and translation (Fig. 10(b)) of the jaw over the stroke, \( \Delta x \), are plotted for the initial (*) and final (o) FE simulations. The compliant mechanism for the initial simulation has model link lengths corresponding to \( k_{1,0} = k_{2,0} = 0 \), and therefore the error trajectory here is entirely attributable to the deformation terms, \( k_{\text{non},0} \), in the model. The polynomial curves fitted to the jaw rotation, \( \omega(\Delta x) \), and \( P_1 \) trajectory, \( y(\Delta x) \), quantify the net deformation-term motion contribution per polynomial order, which informs the designer on how to modify the geometric parameters (i.e., \( \Delta \psi, \Delta g_0, \Delta \phi \)) to minimize the error trajectory. The results from the initial FE simulation (Figs. 10(a) and 10(b), asterisks) indicate the following modifications: increase \( \psi \) with respect to \( \psi \) to redress jaw rotation [16]; decrease \( \psi \) and \( g_0 \) to redress jaw translation errors (Fig. 8). All three geometric parameters may be modified at once.

The influence of these modifications is evaluated by subsequent FE simulation, and iteration continues until the polynomial order of the predicted residual error trajectory becomes apparent. To illustrate the validity of this model, an intermediate step is also shown where only the linear trajectory component of \( P_1 \) (i.e., \( \Delta \psi \)) is corrected, resulting in a parabolic error trajectory (Fig. 10(b), triangles). It is to be noted that the deformation terms, \( k_{\text{non},0} \), include initial link lengths (i.e., \( l_0 \) in Eq. (1a)) as well as extensile components, \( f_0 \), and therefore their magnitude is partially dependent on the undeformed state of the mechanism. This
implies that the trajectory error correction procedure here is inherently iterative because modifying the geometric parameters changes the value of the rigid-body terms and deformation terms.

Fewer than 30 iterations were required to optimize the jaw trajectory, and the final geometric modifications with respect to the initial topology, as denoted in Fig. 9, were:

\[ D_w = 0.850 \text{ deg}, \]
\[ D_g = 0.7 \text{ mm}, \]
\[ D_\psi = 0.354 \text{ deg}. \]

The final compliant gripper exhibits jaw rotation less than 0.8 rad and translation error less than 5 nm over the entire stroke according to the final FE simulation (Figs. 10(a)–10(c); circles). The residual error trajectory of the gripper jaw is 3rd-order, as predicted (Fig. 10(c)). This is therefore the limit of trajectory error correction for the compliant microgripper (Fig. 9).

4 Summary and Discussion

Within the proposed model framework, the error trajectory of a compliant mechanism may generally include both load-independent and load-dependent components (Eq. (6)). The rigid-body terms, \( k_{n,0} \), comprise a load-independent trajectory component, \( \Omega_{n0}(\xi) \), that is attributable to only the mechanism’s topology, irrespective of the mechanism shape. The deformation terms, \( k_{n,\alpha>0} \), comprise a load-dependent component, \( \Omega_{n \alpha} \), that is determined by the mechanism’s topology and shape, as well as the applied load and material properties.

Modification of the mechanism topology alone (i.e., shape remains invariant) may redress trajectory errors up to the \( i \)th order, provided that \( i \)-number of geometric parameters exist within the model framework (i.e., by analysis of the \( k_{n,0} \) terms in Eq. (6)). Having less than \( i \)-number of suitable geometric parameters forces a tradeoff between the values of two or more \( k_{n,0} \) terms in the optimization procedure, regardless of the mechanism shape—and alleviation of this tradeoff would require a different mechanism topology.

Modification of the compliant mechanism’s shape, on the other hand, only affects the magnitude of the load-dependent trajectory component. Within the model, the shape of the segments comprising the compliant mechanism (Figs. 2(b) and 2(c)) affects the extensible components, \( f(\xi) \), but not the initial lengths, \( l_0 \), that define the mechanism topology. Moreover, error trajectories of at least \( (i+1) \)-th order are composed exclusively of deformation terms, \( k_{n,\alpha>0} \), and therefore require compensatory loading to be completely redressed—recall that lumped-compliance shapes may minimize, but not eliminate, load-dependent trajectory components. This is illustrated in the case study, wherein the residual error trajectory of the gripper jaw exhibits a dominant polynomial order of: \( (i+1) = 3 \) (Fig. 10(c)). It is therefore entirely load-dependent and uncorrectable by topology modification alone (i.e., modification of the geometric parameters: \( \Delta \psi, \Delta g, \Delta \delta \)), and will exist irrespective of the compliant mechanism shape.

It is important to realize that the presented model does not assume or impose the motion of a rigid-body kinematic linkage (i.e., as in PRBM). Model construction consists of: (1) identifying particular material points in the compliant mechanism that represent locations of connection between segments comprising
formation terms) are generally nontrivial to calculate, the calculating only noting the existence of significant deformation terms and (2) calculating the trajectory, X composing the trajectory, exhibits a residual 3rd-order error, as predicted. The trajectories in (b,c) are plotted as: X terms of the vectorial distances, l, between these material points. The ability to separate load-independent, X, and load-dependent, X, components in the Taylor series expansion of the trajectory, X, is simply a consequence of admitting the decomposition of Eq. (1). Remarkably, the load-independent trajectory component, X, is equivalent to the trajectory of the rigid-body kinematic linkage corresponding to the undeformed state, X, of the compliant mechanism topology (i.e., as if all f(l) ≡ 0).

While the extensible components, f(l), and therefore the deformation terms are generally nontrivial to calculate, the calculation for the load-independent trajectory, X, is exact and independent of the compliant mechanism shape, as it depends only on the l0 lengths. Herein lies the practical utility of the model because the contributions of topology versus shape to the compliant mechanism’s trajectory may be distinguished by: (1) noting the existence of significant deformation terms and (2) calculating only X. This is illustrated in the case study, wherein analysis and minimization of the gripper’s error trajectory (Fig. 10) required only Eqs. (11) and (14) despite the complicated shape of the microgripper (Fig. 9). The proposed model may also, without difficulty, be extended to three dimensions, as well as to compliant mechanisms with multiple input loads and/or output trajectories.

Last, it is recognized that some ambiguity exists in the selection of material points representing locations of connection which may enable variation, between designers, in the partitioning of a compliant mechanism into segments (Figs. 2(b) and 2(c)); and this variation results from a designer’s interpretation of a compliant mechanism’s topology versus shape. The important point is that the proposed model shows how the configuration of segments—as

Fig. 10 Demonstration of reduced jaw trajectory error in (a) jaw link rotation over the jaw stroke, X, between initial (*) and optimized (0) FE simulations; and (b) Pj, trajectory over jaw stroke between initial (*), 1st-order optimized only (f), and fully optimized (g) FE simulations. (c) The optimized jaw trajectory exhibits a residual 3rd-order error, as predicted. The trajectories in (b,c) are plotted as: X terms of the vectorial distances, l, between these material points. The ability to separate load-independent, X, and load-dependent, X, components in the Taylor series expansion of the trajectory, X, is simply a consequence of admitting the decomposition of Eq. (1). Remarkably, the load-independent trajectory component, X, is equivalent to the trajectory of the rigid-body kinematic linkage corresponding to the undeformed state, X, of the compliant mechanism topology (i.e., as if all f(l) ≡ 0).

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5 Conclusion
We have developed an analytical model for understanding the motion characteristics of an arbitrary planar compliant mechanism. The model is composed of extensible vectorial lengths, l, which represent the configuration of compliant segments comprising the mechanism; whereby the motion trajectory is constructed as an analytical function in terms of these vectorial lengths. Utilizing Taylor series expansion, the trajectory is separated into motion components that are either: (1) load-independent and entirely specifiable by the mechanism topology (i.e., rigid-body terms); or (2) load-dependent and represent all load-geometry interdependencies that arise during mechanism actuation (i.e., deformation terms). A compliant microgripper (Fig. 9) is designed in the case study, which demonstrates the utility of this model for streamlining the compliant mechanism design and trajectory optimization processes, in conjunction FE simulation. As demonstrated in the case study, the model is particularly useful for nonsymmetric mechanism topologies with shapes that are too complex to represent in an analytical closed form.

Acknowledgment
J.B. was supported by a National Science Foundation Graduate Research Fellowship. Additional support to J.B. and A.J.H. was provided by the Office of Naval Research (N000141010556).

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ξ</td>
<td>stroke parameter</td>
</tr>
<tr>
<td>ξ0</td>
<td>stroke parameter at the undeformed state of the mechanism</td>
</tr>
<tr>
<td>ξ − ξ0</td>
<td>actuation range</td>
</tr>
<tr>
<td>l</td>
<td>extensible link length</td>
</tr>
<tr>
<td>l0</td>
<td>initial length of l (corresponding to ξ0)</td>
</tr>
<tr>
<td>f(l)</td>
<td>extensible component of l</td>
</tr>
<tr>
<td>Ωd(ξ)</td>
<td>desired motion trajectory</td>
</tr>
<tr>
<td>Ωc(ξ)</td>
<td>compliant mechanism motion trajectory</td>
</tr>
<tr>
<td>δ(ξ)</td>
<td>error trajectory</td>
</tr>
<tr>
<td>Ωe,dl(ξ)</td>
<td>load-independent component of model motion trajectory</td>
</tr>
<tr>
<td>knd</td>
<td>rigid-body terms (that comprise Ωc,dl(ξ))</td>
</tr>
<tr>
<td>kn,mod</td>
<td>deformation terms (that comprise Ωc,dl(ξ))</td>
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References


