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NONLINEAR STRAIN ENERGY FORMULATION TO CAPTURE THE CONSTRAINT CHARACTERISTICS OF A SPATIAL SYMMETRIC BEAM

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ABSTRACT

In the past, a beam constraint model (BCM) that captures pertinent geometric nonlinearities associated with large displacements has been proposed for slender spatial beams with uniform and symmetric cross-sections. By providing closed-form parametric relations between the end-loads and end-displacements of the beam, the BCM quantifies the constraint characteristics of the beam in terms of stiffness variations, parasitic error motions, and the cross-axis coupling. This paper presents a nonlinear strain and strain energy formulation for the spatial symmetric beam, based on assumptions that are consistent with the BCM. This strain energy derivation, employing the Principle of Virtual Work, provides a simpler mathematical approach for the analysis of flexure mechanisms with multiple spatial beams. Using this formulation, we obtain the stiffness relations in the transverse bending directions, the constraint relations in the axial and torsional directions, and the overall strain energy expression in terms of the beam end-loads and end-displacements. These expressions, collectively the BCM, are in form that is suitable for the analysis of multi-beam flexure mechanisms.

1. INTRODUCTION AND BACKGROUND

Flexure mechanisms provide guided motion via elastic deformation and are used in a variety of applications that demand high precision, minimal assembly, long operating life, and/or design simplicity [1, 2]. The spatial beam flexure, sometimes also referred to as a wire flexure, is commonly used as a constraint element in the design of flexure mechanisms [3, 4]. The constraint behavior of a spatial beam is demonstrated in Fig.1. Due to the slenderness of the beam in the Y and Z directions, the stiffness values associated with bending in the XY and XZ planes and the torsion about the X axis are relatively low. On the other hand, the translational stiffness along the X axis is relatively high.

Given this contrast in stiffness, the slender spatial beam serves as a constraint element in terms of its end-displacements with respect to a reference ground – it constrains motion along the U_{XL} translation, and allows motion along the U_{YL} and U_{ZL} translations and Θ_{XL}, Θ_{YL} and Θ_{ZL} rotations. As is common for flexure-based constraint elements [5], the terms degree of constraint (DoC) and degree of freedom (DoF) are used here to refer to the *stiff* and *compliant* motion directions, respectively.

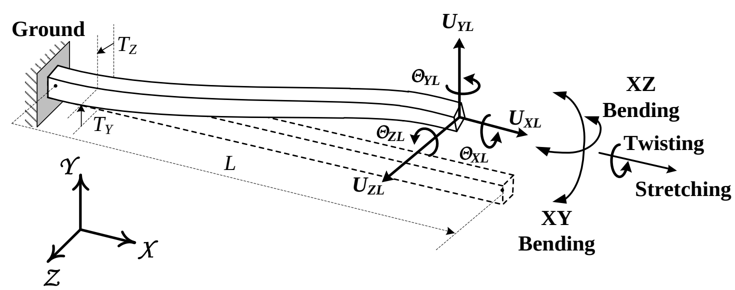

Fig.1 Spatial Beam Flexure: Undeformed and Deformed

To accurately predict the motion performance of a three-dimensional flexure mechanism that comprises one or more spatial beams, it is important to first understand, qualitatively and quantitatively, the constraint characteristics of the individual spatial beam. Of particular interest is the stiffness along each of the six motion directions associated with the end of the beam and its variation with increasing end-forces and end-displacements. It is also important to identify the error motions, which by definition are undesired motions. These may be categorized as cross-axis errors (motion in a DoF direction due to displacement in another DoF direction) and parasitic error (motion in a DoC direction) [5, 6].

Previous analytical and experimental results have shown that geometric nonlinearities strongly influence the above-mentioned stiffness behavior and error motions in beam

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flexures [5-8] undergoing large displacements. Using an explicit Newtonian approach, these effects have been accurately captured for a spatial, uniform, symmetric[1] beam in the beam constraint model (BCM), which comprises closed-form parametric relations between the end-loads and end-displacements [8]. This approach, although mathematically accurate, proves to be tedious while analyzing multi-beam flexure mechanisms. A well-known alternative is to use energy methods [9], which avoid the unnecessary computation of internal loads and deliver the force-displacement relations at the point of interest in the overall mechanism.

However, to apply an energy method such as the Principle of Virtual Work (PVW), an accurate, closed-form strain energy expression for the beam in terms of its end-displacements is required, which serves as the motivation for this paper. A non-linear strain and strain energy derivation for a uniform, symmetric beam is presented in *Section 2*, with emphasis on capturing the relevant nonlinearities and recognizing appropriate approximations that are consistent with the previous explicit formulation. In *Section 3*, we employ the above strain energy expression in the PVW to derive the governing equations of the spatial beam. These include the differential equations that govern the bending, torsion, and stretching of the beam, along with natural boundary conditions. In *Section 4*, we derive a closed-form and parametric solution to the above differential equations and boundary conditions to obtain the transverse bending stiffness relations, the geometric relations in the axial and torsional directions, and the overall strain energy relation in terms of the beam end-loads and end-displacements. A consistent truncation scheme is proposed to further simplify the final form of these expressions and render them suitable for the closed-form analysis of flexure mechanisms made of multiple spatial beams. We conclude in *Section 5* with a brief summary of contributions and plans for future work.

2. NONLINEAR STRAIN AND STRAIN ENERGY FORMULATION FOR A SPATIAL BEAM

In order to determine the nonlinear strain, the spatial deformation of the beam needs to be mathematically characterized. When a long, slender[2], circular cross-section beam is subjected to pure bending and torsion, symmetry implies that the Euler-Bernoulli assumption holds true [10], i.e. plane sections remain plane and perpendicular to the neutral axis after deformation. However, in the case of pure bending and torsion of long, slender rectangular beams, small warping of cross-section does take place in order to satisfy boundary conditions for shearing stresses [10]. In spite of this, for displacement (U_{YL} and U_{ZL}) in the range of 0.1L, where L is the length of the spatial beam, and rotations (Θ_{XL}, Θ_{YL}, and Θ_{ZL}) in the range of 0.1 rad, it can be argued for the beam shown in Fig.1 that these effects can be approximately superimposed. In other words, an initially plane cross-section first undergoes a rigid-body translation and rotation to remain plane and perpendicular to the neutral axis, followed by a small cross-sectional warp (Fig.2). The rigid-body rotation is separately shown in Fig.2 via the Euler angles α, β and Θ_{Xd}. It should be noted here that the twist Θ_{Xd}, is defined in the deformed coordinate frame X_d–Y_d–Z_d, rather than the more intuitive undeformed X–Y–Z coordinate frame. The deformed X_d–Y_d–Z_d coordinate frame actually varies with coordinate X but at any particular point has its X_d axis tangential to the deformed neutral axis of the beam while Y_d and Z_d form the axis of the principle moment of area of the cross-section in the deformed state.

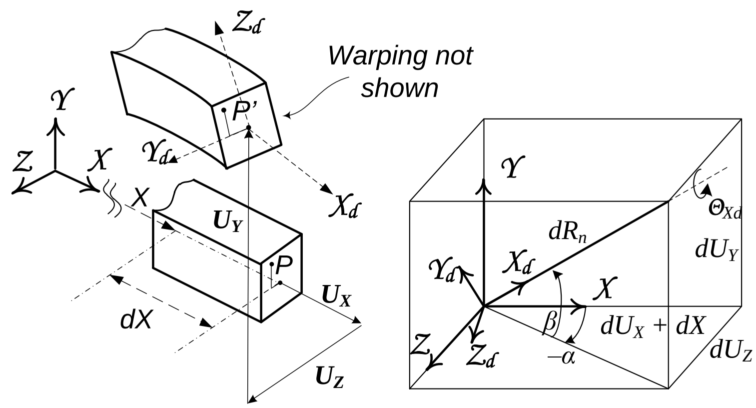

Fig.2 Spatial Kinematics of Beam Deformation

Saint Venant's solution for a long slender beam with square cross-section [10] under pure torsion shows that warping causes only parallel motion of points on any cross-section with respect to the neutral axis of the beam; in other words, in-plane distortion is absent. Additionally, the analytical closed-form warping function of the cross-section is found to be constant along the length of the beam i.e. independent of the X coordinate. This implies that the point P, shown in Fig.2, upon deformation will have the same Y and Z coordinates in the deformed coordinate system, X_d–Y_d–Z_d, but will also have a small out of plane motion (not shown in the figure) dependent only on its Y and Z coordinates. Even though in the presence of transverse forces and moments or for cross-sections close to the beam ends, this solution is not strictly true, previous studies [10, 11] confirm that warping remains largely parallel to neutral axis and constant along the length of the beam, for end displacements within 10% of the beam length.

Furthermore, Da Silva [12] and Hodges [13] showed that the warping in a slender beam is small and its effect can be dropped in the axial strain ε_{XX} in comparison to larger effects of bending and axial stretching. However, warping was shown to have a non-negligible effect on shear strains associated with

[1] Uniform implies a non-varying cross-section along the beam length. Symmetric implies equal moments of area of the beam cross-section about the Y and Z axes.
[2] Slender generally implies a length to thickness ratio greater than 20 [10, 11]

torsion. This contribution of warping can be easily captured in the torsional moment of area.

With this qualitative understanding of the deformation of the beam, we next proceed to quantitatively determine the strain at any general point P with coordinate position (X, Y, Z). As shown in Fig.2, U_X, U_Y and U_Z, are defined in the X–Y–Z coordinate frame, and describe the rigid body translation of the centroid of a cross-sectional area. These translations along with the rotational displacements α, β and Θ_{Xd}, all of which are functions of the X coordinate location of the cross-section, form a set of six coordinates to describe the deformed position of point P and hence the strain.

The Green's strain measure [14] is used to determine the strain components in terms of the undeformed coordinate system. However, since the maximum end displacements and rotations are limited to 0.1L and 0.1 radians, higher order nonlinear terms that are more than two orders of magnitude smaller than the primary bending and torsion effects become insignificant. For the purpose of obtaining a solution that is correct to the second order[3], these terms are dropped.

Using the above assumptions, the final strain expression are given by Eq. (1) and (2) below. The detailed derivation of this strain can be found in the references [8, 12, 13].

$$\begin{aligned}\varepsilon_{XX} &= U'_X + \frac{1}{2}U'^2_Y + \frac{1}{2}U'^2_Z - Y\kappa_{Zd} + Z\kappa_{Yd} \\ &\quad + \frac{1}{2}\kappa_{Xd}{}^2\left(Y^2 + Z^2\right)\end{aligned} \tag{1}$$

$$\begin{aligned}\gamma_{xy} &= \kappa_{Xd}\left[Y - Y_W\right] \\ \gamma_{xz} &= \kappa_{Xd}\left[Z - Z_W\right]\end{aligned} \tag{2}$$

It should be noted here that although finite end displacements are considered, the strains are still small because the beam is assumed to be slender. The first three terms in the axial strain, ε_{XX}, collectively represent the elastic stretching in the axial direction, while correcting for kinematic effects. The next three terms depend on the beam curvatures κ_{Xd}, κ_{Yd} and κ_{Zd}, which are defined in the deformed coordinate axis X_d–Y_d–Z_d. These terms arise from the combined effect of torsion and bending and depend only on X. Although the last of these three terms is significantly smaller than the other terms, it is retained because it becomes significant in the absence of axial stretching and bending loads. The approximate value of the three beam curvatures, accurate to the second order are given below. A detailed derivation of the following simplified expressions can be found in previous work [8, 13].

$$\begin{aligned}\kappa_{Xd} &\approx \Theta'_{Xd} - U''_Z U'_Y \\ \kappa_{Yd} &\approx \sin\left(\Theta_{Xd}\right)U''_Y - \cos\left(\Theta_{Xd}\right)U'_Z \\ \kappa_{Zd} &\approx \cos\left(\Theta_{Xd}\right)U''_Y + \sin\left(\Theta_{Xd}\right)U''_Z\end{aligned} \tag{3}$$

The shear strains given in Eq.(2) depend on curvature κ_{Xd} and warping. However, since the effect of warping is assumed to be small, it can be factored using the correction terms Y_W and Z_W [12]. Strains ε_{YY} and ε_{ZZ} are also present due to Poisson's effect. However, shear strain γ_{YZ} is zero due to the absence of in-plane distortion of the cross-section. Other nonlinear terms in strain expressions reported in the previous literature [12, 13, 15] are at most of the order of 10^{-5} and contribute negligibly to the strains, which are generally of the order of 10^{-2} for the given maximum loading conditions. Therefore, these nonlinear terms have been dropped in Eq.(1) and (2). It should be noted here that infinitesimal strain theory does not capture the $\kappa_{Xd}{}^2$ term in ε_{XX} or warping effect in the shear strains γ_{XY} and γ_{XZ} and hence not been used in this formulation.

Using the strain expressions in Eq.(1) and (2) and assuming linear material properties, the strain energy for the spatial beam flexure may be expressed as follows:

$$\begin{aligned}V &= \frac{E}{2}\int_{vol}\varepsilon^2_{xx}\,dAdX + \frac{G}{2}\int_{vol}\left(\gamma^2_{xy} + \gamma^2_{xz}\right)dAdX \\ &= V_1 + V_2\end{aligned} \tag{4}$$

As seen here, there are two components of the strain energy: V_1 is the strain energy due to ε_{XX}, which arises from transverse bending and axial stretching, and V_2 is the energy due to the shear strains, which arise due to torsion of the beam.

The strain energy contribution from the strains ε_{YY} and ε_{ZZ} is zero for the following reasons. Due to the slenderness of the beam, the variation of stresses σ_{YY} and σ_{ZZ} in the Y and Z directions, respectively, can be argued to be equal to zero which means σ_{YY} and σ_{ZZ} are constants. However, in the absence of transverse surface loading in the Y and Z direction, the only constant value of σ_{YY} and σ_{ZZ} possible is zero. Therefore, even though ε_{YY} and ε_{ZZ} are finite due to Poisson's effect, and are equal to $-\nu\varepsilon_{XX}$, where ν is Poisson's ratio, the associated stresses in these directions are negligible. Hence ε_{YY} and ε_{ZZ} can be neglected in the strain energy calculation. Furthermore, using the elemental equilibrium conditions in Eq.(5), it can be shown that in the present case of zero σ_{YY}, σ_{ZZ} and τ_{YZ}, κ_{Xd} is constant with respect to X.

$$\frac{\partial\tau_{YX}}{\partial X} + \frac{\partial\sigma_{YY}}{\partial Y} = 0 \;\Rightarrow\; \frac{\partial\tau_{YX}}{\partial X} = 0 \;\Rightarrow\; \frac{\partial\kappa_{Xd}}{\partial X} = 0 \tag{5}$$

Moving ahead, the two parts of strain energy from Eq.(4) may be expanded using the strain expressions from Eq.(1) and (2)

$$\begin{aligned}V_1 = &\frac{E}{2}\int_{vol}\left(U'_X + \frac{1}{2}U'^2_Y + \frac{1}{2}U'^2_Z\right)^2 dAdX \\ &+ \frac{E}{2}\int_{vol}2\left(U'_X + \frac{1}{2}U'^2_Y + \frac{1}{2}U'^2_Z\right)\left(-Y\kappa_{Zd} + Z\kappa_{Yd}\right)dAdX \\ &+ \frac{E}{2}\int_{vol}\left\{\kappa_{Xd}{}^2\left(Y^2 + Z^2\right)\right\}\left(-Y\kappa_{Zd} + Z\kappa_{Yd}\right)dAdX \\ &+ \frac{E}{2}\int_{v}\left(-Y\kappa_{Zd} + Z\kappa_{Yd}\right)^2 dAdX \\ &+ \frac{E}{2}\int_{vol}\left(U'_X + \frac{1}{2}U'^2_Y + \frac{1}{2}U'^2_Z\right)\left[\kappa_{Xd}{}^2\left(Y^2 + Z^2\right)\right]dAdX\end{aligned}$$

[3] Accurate to the second order implies that the error is less than 1%.

$$+\frac{E}{8}\int_{vol}\left\{\kappa_{Xd}{}^{2}\left(Y^{2}+Z^{2}\right)\right\}^{2} dAdX = I_1+I_2+I_3+I_4+I_5+I_6$$

$$V_2=\frac{G}{2}\int_{vol}\kappa_{Xd}{}^{2}\left(Y_e^2+Z_e^2\right)dAdX = I_7$$

where $Y-Y_W \triangleq Y_e\ , Z-Z_W \triangleq Z_e$ (6)

In the first part of the energy, V_1, the six integrals are serially denoted by I_1 through I_6. Of these, the integrals I_2 and I_3 are zero by the definition of the neutral axis that passes through the centroid of the cross-section. Integral I_6 is also dropped as it is at least four orders of magnitude smaller than integral I_1 due to the slenderness of the beam and the twisting angle Θ_{Xd} being limited to ±0.1 radians.

Next, the strain energy expression is simplified by recognizing that the beam curvatures, given in Eq.(3), are only dependent on the axial coordinate X. Thus, the volume integral can be decomposed into a double integral over the cross-sectional area and a single integral over the axial coordinate X. This results in the following simplified expression.

$$\begin{aligned} V &= \frac{EA}{2}\int_0^L\left(U'_X+\frac{1}{2}U'^2_Y+\frac{1}{2}U'^2_Z\right)^2 dX+\frac{EI}{2}\int_0^L\left(U''^2_Y+U''^2_Z\right)dX \\ &\quad +EI\int_0^L\left(U'_X+\frac{1}{2}U'^2_Y+\frac{1}{2}U'^2_Z\right)\kappa^2_{Xd}dX+\frac{GJ}{2}\int_0^L\kappa^2_{Xd}dX \qquad (7)\\ &= I_1+I_4+I_5+I_7 \end{aligned}$$

The first integral I_1 in V_1 (Eq.(7)) describes energy associated with axial stretching. Through U_Y' and U_Z', it also captures the coupling between the bending directions and the axial constraint direction. The second term, I_4, captures the energy that originates from bending. The third term, I_5, captures the coupling between the torsion and axial stretching directions. Finally, the last term I_7 captures the energy from pure torsion.

It should be noted here that in the last step of deriving Eq.(7), we have also assumed a symmetric beam cross-section, which implies that the two principal bending moments of area (I_{YY} and I_{ZZ}) are identical and equal to I. Also, due to this symmetry, the polar moment of area is equal to $2I$. The torsion constant is defined as J and is, in general, different from the polar moment of area due to warping [10, 11]. These assumptions may be expressed as follows:

$$\text{where } \int_A Y^2 dA=\int_A Z^2 dA \triangleq I,\quad \int_A YZdA=0$$
$$\int_A\left(Y^2+Z^2\right)dA=2I,\quad \int_A\left(Y_e^2+Z_e^2\right)dA\triangleq J \qquad (8)$$

3. GOVERNING EQUATIONS FOR A SPATIAL SYMMTRIC BEAM

The Principle of Virtual Work (PVW) dictates that for an elastic body in static equilibrium, the virtual work done by external forces over a set of geometrically compatible but otherwise arbitrary 'virtual' displacements is equal to the change in the strain energy of this body due to these 'virtual' displacements [9]. This is mathematically expressed as:

$$\delta W=\delta V \qquad (9)$$

Therefore, the first step in applying PVW would be to determine the strain energy of the beam (Eq.(7)), with respect to a variation of its displacements. For the following procedure, we choose U_X, U_Y, U_Z, Θ_{Xd}, U'_Y and U'_Z to be the generalized coordinates which, along with their boundary conditions, completely define the displacement of the beam.

For the sake of clarity, we consider the variation of the four integrals in Eq.(7) one at a time.

Let $\left(U'_X+\frac{1}{2}U'^2_Y+\frac{1}{2}U'^2_Z\right)\triangleq \bar{U}'_x$

Then,

$$\delta I_1 = EA\int_0^L \bar{U}'_X\left(\frac{d\left(\delta U_X\right)}{dX}+U'_Y\frac{d\left(\delta U_Y\right)}{dX}+U'_Z\frac{d\left(\delta U_Z\right)}{dX}\right)dX$$
$$= EA\left\{\bar{U}'_X\left(\delta U_X+U'_Y\,\delta U_Y+U'_Z\,\delta U_Z\right)\right\}\Big|_0^L - EA\int_0^L \bar{U}''_X\delta U_X\,dX$$
$$-EA\int_0^L\left(\bar{U}'_X U'_Y\right)'\delta U_Y dX - EA\int_0^L\left(\bar{U}'_X U'_Z\right)'\delta U_Z dX$$

$$\delta I_4=\frac{EI}{2}\int_0^L\delta\left(U''^2_Y+U''^2_Z\right)dX = EI\int_0^L\left(U''_Y\delta U''_Y+U''_Z\delta U''_Z\right)dX$$
$$= EI\left(U''_Y\delta U'_Y+U''_Z\delta U'_Z-U'''_Y\delta U_Y-U'''_Z\delta U_Z\right)\Big|_0^L$$
$$+EI\int_0^L\left(U^{iv}_Y\delta U_Y+U^{iv}_Z\delta U_Z\right)dX$$

$$\delta I_5 = EI\int_0^L\delta\left(\bar{U}'_X\right)\kappa^2_{Xd}dX+EI\int_0^L\bar{U}'_X\delta\left(\kappa^2_{Xd}\right)dX$$
$$= EI\int_0^L\left(\frac{d\left(\delta U_X\right)}{dX}+U'_Y\frac{d\left(\delta U_Y\right)}{dX}+U'_Z\frac{d\left(\delta U_Z\right)}{dX}\right)\kappa^2_{Xd}dX$$
$$+EI\int_0^L\bar{U}'_X\delta\left(\kappa^2_{Xd}\right)dX$$
$$= EI\left\{\kappa^2_{Xd}\left(\delta U_X+U'_Y\,\delta U_Y+U'_Z\,\delta U_Z\right)\right\}\Big|_0^L - EI\int_0^L\left(\kappa^2_{Xd}\right)'\delta U_X\,dX$$
$$-EI\int_0^L\left(\kappa^2_{Xd}U'_Y\right)'\delta U_Y dX - EI\int_0^L\left(\kappa^2_{Xd}U'_Z\right)'\delta U_Z dX$$
$$+2EI\left[\bar{U}'_X\kappa_{Xd}\delta\Theta_{Xd}-\bar{U}'_X\kappa_{Xd}\left(U'_Y\,\delta U'_Z+U''_Z\,\delta U_Y\right)+\left(\bar{U}'_X\kappa_{Xd}U'_Y\right)'\delta U_Z\right]\Big|_0^L$$
$$-2EI\left[\int_0^L\left\{\left(\bar{U}'_X\kappa_{Xd}U'_Y\right)''\delta U_Z-\left(\bar{U}'_X\kappa_{Xd}U''_Z\right)'\delta U_Y\right\}dX\right]$$
$$-2EI\int_0^L\left\{\kappa_{Xd}\bar{U}'_X\right\}'\delta\Theta_{Xd}dX$$

$$\delta I_7 = GJ\left[\left\{\kappa_{Xd}\delta\Theta_{Xd} - \kappa_{Xd}\left(U_Y'\,\delta U_Z' + U_Z''\,\delta U_Y\right) + \left(\kappa_{Xd}U_Y'\right)'\delta U_Z\right\}\right]_0^L$$
$$-GJ\left[\int_0^L\left\{\left(\kappa_{Xd}U_Y'\right)''\delta U_Z - \left(\kappa_{Xd}U_Z''\right)'\delta U_Y\right\}dX\right] - GJ\int_0^L \kappa_{Xd}'\delta\Theta_{Xd}\,dX \tag{10}$$

Based on Eq.(10), it is evident that the variation of the strain energy may be expressed in terms of the six generalized virtual displacements δU_X, δU_Y, δU_Z, $\delta\Theta_{Xd}$, $\delta U_Y'$ and $\delta U_Z'$, which are all variables in the X coordinate, along with their boundary values at X = 0 and L.

At the fixed end i.e. X=0

$$\delta U_X = 0;\ \delta U_Y = 0;\ \delta U_Z = 0;\ \delta\Theta_{Xd} = 0;\ \delta\Theta_Y = 0;\ \delta\Theta_Z = 0 \tag{11}$$

At the free end i.e. X=L

$$\begin{aligned}&\delta U_X = \delta U_{XL};\ \delta U_Y = \delta U_{YL};\ \delta U_Z = \delta U_{ZL};\\ &\delta\Theta_{Xd} = \delta\Theta_{XdL};\ \delta\Theta_Y = \delta\Theta_{YL};\ \delta\Theta_Z = \delta\Theta_{ZL}\end{aligned} \tag{12}$$

Thus, the variation of the total strain energy in terms of the above virtual displacements provides the right hand side of Eq.(9) .

Given the external loads $\boldsymbol{F_{XL}}$, $\boldsymbol{F_{YL}}$, $\boldsymbol{F_{ZL}}$, $\boldsymbol{M_{XL}}$, $\boldsymbol{M_{YL}}$ and $\boldsymbol{M_{ZL}}$, the virtual work on the left hand side of Eq.(9), may be expressed as

$$\begin{aligned}\delta W = {}&\boldsymbol{F_{XL}}\delta U_{XL} + \boldsymbol{F_{YL}}\delta U_{YL} + \boldsymbol{F_{ZL}}\delta U_{ZL} + \boldsymbol{M_{XL}}\delta\Theta_{XL}\\ &+ \boldsymbol{M_{YL}}\delta\Theta_{YL} + \boldsymbol{M_{ZL}}\delta\Theta_{ZL}\end{aligned} \tag{13}$$

where δU_{XL}, δU_{YL}, δU_{ZL}, $\delta\Theta_{XL}$, $\delta\Theta_{YL}$ and $\delta\Theta_{ZL}$ represent six independent virtual displacements at the beam end, expressed in the direction of the external loads.

For the application of PVW, these six virtual end displacements in Eq.(13) have to be expressed in terms of the previous set of six virtual end displacements given in Eq.(10). This requires expressing $\delta\Theta_{XL}$, $\delta\Theta_{YL}$ and $\delta\Theta_{ZL}$ as a function of δU_{XL}, δU_{YL}, δU_{ZL}, $\delta\Theta_{XdL}$, $\delta U_{YL}'$ and $\delta U_{ZL}'$. This is done by recognizing the fact that the virtual rotations can be chosen to be arbitrarily small, and therefore be represented as vectors.

Referring to Fig.2, since the final orientation of the X_d–Y_d–Z_d coordinate frame is unique; the virtual rotations may be expressed as variations of the Euler angles:

$$\begin{aligned}&\delta\theta_{XL}\hat{i} + \delta\theta_{YL}\hat{j} + \delta\theta_{ZL}\hat{k} = -\delta\alpha\,\hat{j}\Big|_L + \left\{\cos(\alpha)\hat{k} - \sin(\alpha)\hat{i}\right\}\delta\beta\Big|_L\\ &+\left\{\frac{1+U_X'}{1+\bar{U}_X'}\hat{i} + \frac{U_Y'}{1+\bar{U}_X'}\hat{j} + \frac{U_Z'}{1+\bar{U}_X'}\hat{k}\right\}\delta\Theta_{Xd}\Bigg|_L\end{aligned} \tag{14}$$

Furthermore, using the geometry shown in Fig.2 the variations of Euler angles α and β can be expressed in terms of U_{XL}', U_{YL}', U_{ZL}', $\delta U_{YL}'$, and $\delta U_{ZL}'$ as follows

$$\begin{aligned}&\delta\alpha = -\frac{\delta U_Z'}{1+U_X'} + \frac{U_Z'\delta U_X'}{\left(1+U_X'\right)^2}\\ &\delta\beta = \frac{\delta U_Y'}{1+U_X' + \frac{1}{2}U_Z'^2} - \frac{U_Y'\left(\delta U_X' + U_Z'\delta U_Z'\right)}{\left(1+U_X' + \frac{1}{2}U_Z'^2\right)^2}\end{aligned} \tag{15}$$

For the range of end displacements considered $\bar{U}_{XL}'$, U_{XL}', U_{YL}', and U_{ZL}' are of the order of 10^{-2}, 10^{-2}, 10^{-1} and 10^{-1} respectively. Therefore, suitable second order approximations are made to simplify Eq.(14) to yield:

$$\begin{aligned}&\delta\Theta_{XL} \approx \delta\Theta_{XdL} - U_{ZL}'\delta U_{YL}' + U_{ZL}'U_{YL}'\left(\delta U_{XL}' + U_{ZL}'\delta U_{ZL}'\right)\\ &\delta\Theta_{YL} \approx -\delta U_{ZL}' + U_{ZL}'\delta U_{XL}' + U_{YL}'\delta\Theta_{XdL}\\ &\delta\Theta_{ZL} \approx \delta U_{YL}' - U_{YL}'\left(\delta U_{XL}' + U_{ZL}'\delta U_{ZL}'\right) + U_{ZL}'\delta\Theta_{XdL}\end{aligned} \tag{16}$$

Using Eq.(16), the left hand side of PVW in Eq.(9) can be expressed in terms of $\delta U_{XL}, \delta U_{YL}, \delta U_{ZL}, \delta\Theta_{XdL}, \delta U_{YL}'$ and $\delta U_{ZL}'$ as is the right hand side. As stated in PVW, these virtual displacements are geometrically consistent but otherwise arbitrary. Hence, for Eq.(9) to hold, the respective coefficients of the virtual displacements must be identical and are equated to get the desired beam governing differential equations and natural boundary conditions. It should be noted here that the left hand side of the PVW also has the term U_{XL}'. Given that there can be only six independent generalized displacements, U_{XL}' is not an independent variable. This poses a problem in comparing coefficients as the relation of U_{XL}' to other variables is still unknown. However, U_{XL}' is definitely not dependent on U_{XL} due to the differentiation with respect to the X, the independent variable of the problem. Hence, the coefficient of δU_{XL} and δU_X are compared first.

$$\left(EA\bar{U}_X' + EI\kappa_{Xd}^2\right)\Big|_L = \boldsymbol{F_{XL}},\ \text{and}$$
$$\left(EA\bar{U}_X' + EI\kappa_{Xd}^2\right)' = 0$$
$$\Rightarrow EA\bar{U}_X' + EI\kappa_{Xd}^2 = \text{constant} = \boldsymbol{F_{XL}} \tag{17}$$

Since Eq.(5) shows κ_{Xd} to be constant in X, the variation of $\bar{U}_X'$ calculated from Eq.(17), is found to be zero. Hence, the dependence of $\delta U_{XL}'$ on the variations of the six generalized displacement variables is determined as

$$\delta U_X' = -U_Y'\delta U_Y' - U_Z'\delta U_Z' \tag{18}$$

The value of $\delta U_{XL}'$ is substituted in Eq.(16) and the resulting equation corrected to the second order is given below.

$$\begin{aligned}&\delta\Theta_{XL} = \delta\Theta_{XdL} - U_{ZL}'\delta U_{YL}'\\ &\delta\Theta_{YL} = -\delta U_{ZL}' - U_{YL}'U_{ZL}'\delta U_{YL}' + U_{YL}'\delta\Theta_{XdL}\\ &\delta\Theta_{ZL} = \delta U_{YL}' + U_{ZL}'\delta\Theta_{XdL}\end{aligned} \tag{19}$$

One can now start comparing the coefficient of virtual displacements on the left and right hand sides of Eq.(9). Comparing the coefficients of $\delta\Theta_{XdL}$ and $\delta\Theta_{Xd}$ we get

$$\left\{GJ\left(1+\frac{2EI}{GJ}\bar{U}_X'\right)\kappa_{Xd}\right\}\Bigg|_L = \boldsymbol{M_{XdL}}\ \text{and}$$
$$\left\{\left(1+\frac{2EI}{GJ}\bar{U}_X'\right)\kappa_{Xd}\right\}' = 0$$
$$\Rightarrow \kappa_{Xd} = \left\{\Theta_{Xd}' - U_Z''\,U_Y'\right\} = \text{constant} \approx \frac{\boldsymbol{M_{XdL}}}{GJ}\left(1-\frac{2EI}{GJ}\bar{U}_X'\right) \tag{20}$$

Eq.(17) and Eq.(20) can be approximately solved as

$$U'_{XL} + \frac{1}{2}U'^2_{YL} + \frac{1}{2}U'^2_{ZL} \approx \frac{\boldsymbol{F_{XL}}}{EA} - \frac{I\boldsymbol{M^2_{XdL}}}{G^2J^2A} \tag{21}$$

$$\left\{\Theta'_{Xd} - U''_Z\, U'_Y\right\} \approx \frac{\boldsymbol{M_{XdL}}}{GJ} - \frac{2I\boldsymbol{M_{XdL}F_{XL}}}{G^2J^2A} \tag{22}$$

It should be noted here that although solving for $\bar{U}'_X$ from Eq.(17) and (20) requires solving a quadratic equation, for twisting angle Θ_{Xd} less than 0.1 radians, the coefficient of the quadratic term is relatively small and therefore can be dropped. Equation (21) and (22) are the characteristic differential equations associated with axial stretching and torsion respectively. Second order approximations made in deriving these equations are justified because $\bar{U}'_X$ and κ_{Xd} being of the order of 0.01 and 0.1 respectively.

Equating the coefficients of the remaining virtual displacements, δU_{YL}, δU_Y, δU_{ZL}, δU_Z, $\delta U'_{YL}$, and $\delta U'_{ZL}$, two more characteristic differential equations associated with bending in the XY and XZ planes are obtained along with four natural boundary conditions.

$$\begin{aligned} EIU_Y^{iv} - \boldsymbol{F_{XL}}U''_Y + \boldsymbol{M_{XdL}}U'''_Z &= 0 \\ EIU_Z^{iv} - \boldsymbol{F_{XL}}U''_Z - \boldsymbol{M_{XdL}}U'''_Z &= 0 \end{aligned} \tag{23}$$

Natural Boundary Conditions:

$$\begin{aligned} F_{YL} &= F_{XL}U'_{YL} - EIU'''_{YL} - M_{XdL}U''_{ZL} \\ F_{ZL} &= F_{XL}U'_{ZL} - EIU'''_{ZL} + M_{XdL}U''_{YL} \\ M_{YL} &= -EIU''_{ZL} + M_{XdL}U'_{YL} \\ \left(1+U'^2_{ZL}\right)M_{ZL} &= EIU''_{YL} + M_{XdL}U'_{ZL} \\ \Rightarrow M_{ZL} &\approx EIU''_{YL} + M_{XdL}U'_{ZL} \end{aligned} \tag{24}$$

The final approximation, although consistent with previous second order approximation, is not necessary from the point of view of the beam mechanics, as the boundary conditions need not be symmetric. However, with this approximation the final model is more simple and easy to use. Using this final set of beam governing equations (21), (22) and (23) along with the natural boundary conditions in Eq.(24) and the geometric boundary conditions in Eq.(11) and (12), the closed form energy model and constraint conditions of the spatial beam will be derived in the next section.

It should be noted here that the beam characteristic differential equation derived here is consistent with previously derived more accurate but complex nonlinear beam models [12, 13], when subjected to the same assumptions and second order approximations that have been made here.

Compared to a linear analysis, the governing Eqs.(21), (22) and (23) take into account many nonlinear effects. In Eq. (21) and (22) the kinematic effect of bending and the elastic coupling effect of torsion and axial stretching is captured in addition to linear stretching and twisting. Eq. (23) captures the effects of axial force and moment on bending which is not captured in linear bending analysis. Although capturing these effects renders the governing equations of stretching and torsion to be nonlinear, the bending equation is still linear in U_Y and U_Z. This allows Eq.(23) to be solved with relative ease and then the results can be substituted in Eq. (21) and (22) in order to find a solution of U_X and θ_{Xd}.

4. NONLINEAR STRAIN ENERGY IN TERMS OF END-LOADS AND END-DISPLACEMENTS

The solution to the beam characteristic differential equations (21), (22) and (23) is obtained next in order to find a closed form parametric strain energy expression. At this point in the analysis, we proceed to normalize all the loads and displacements per the following scheme:

$$\begin{aligned} &\boldsymbol{m_{z1}} \triangleq \frac{\boldsymbol{M_{ZL}}L}{EI}, \quad \boldsymbol{m_{y1}} \triangleq \frac{\boldsymbol{M_{YL}}L}{EI}, \quad \boldsymbol{m_{xd1}} \triangleq \frac{\boldsymbol{M_{XdL}}L}{EI}, \quad \boldsymbol{f_{z1}} \triangleq \frac{\boldsymbol{F_{ZL}}L^2}{EI}, \\ &v \triangleq \frac{VL}{EI}, \quad \boldsymbol{f_{y1}} \triangleq \frac{\boldsymbol{F_{YL}}L^2}{EI}, \quad \boldsymbol{f_{x1}} \triangleq \frac{\boldsymbol{F_{XL}}L^2}{EI}, \quad u_y \triangleq \frac{U_Y}{L}, \quad u_z \triangleq \frac{U_Z}{L}, \\ &u_{y1} \triangleq \frac{U_{YL}}{L}, \quad u_{z1} \triangleq \frac{U_{ZL}}{L}, \quad x \triangleq \frac{X}{L}, \quad \theta_{xd} \triangleq \Theta_{Xd}, \quad \theta_{xd1} \triangleq \Theta_{XdL} \end{aligned} \tag{25}$$

Using this normalization scheme, Eq.(23) can be represented in the matrix form given below.

$$\begin{Bmatrix} u'''_y \\ u''_y \\ u'''_z \\ u''_z \end{Bmatrix}' = \begin{bmatrix} 0 & \boldsymbol{f_{x1}} & -\boldsymbol{m_{xd1}} & 0 \\ 1 & 0 & 0 & 0 \\ \boldsymbol{m_{xd1}} & 0 & 0 & \boldsymbol{f_{x1}} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u'''_y \\ u''_y \\ u'''_z \\ u''_z \end{Bmatrix} \tag{26}$$

The four scalar equations represented above can be solved by first decoupling them. This may be done by determining the eigenvalues and eigenvectors of the square matrix in the above equation.

$$\text{E-values:} \quad \lambda_1 = \lambda, \quad \lambda_2 = -\lambda, \quad \lambda_3 = \frac{\boldsymbol{f_{x1}}}{\lambda}, \quad \lambda_4 = -\frac{\boldsymbol{f_{x1}}}{\lambda}$$

$$\text{where,} \quad \lambda \triangleq \frac{1}{2}\sqrt{4\boldsymbol{f_{x1}} - 2\boldsymbol{m_{xd1}}^2 + 2\boldsymbol{m_{xd1}}\sqrt{\boldsymbol{m_{xd1}}^2 - 4\boldsymbol{f_{x1}}}}$$

$$\text{E-vector matrix:} \quad [Q] = \begin{bmatrix} -\boldsymbol{r} & -\boldsymbol{r} & -\dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r}} & -\dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r}} \\ -\dfrac{\boldsymbol{r}}{\lambda_1} & -\dfrac{\boldsymbol{r}}{\lambda_2} & -\dfrac{\boldsymbol{f_{x1}}}{\lambda_3 \boldsymbol{r}} & -\dfrac{\boldsymbol{f_{x1}}}{\lambda_4 \boldsymbol{r}} \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{where,} \quad \boldsymbol{r} \triangleq \frac{1}{2}\left[\boldsymbol{m_{xd1}} - \sqrt{\boldsymbol{m_{xd1}}^2 - 4\boldsymbol{f_{x1}}}\right]$$

The eigenvalues λ_1, λ_2, λ_3 and λ_4 are distinct[4] for non-zero $\boldsymbol{f_{x1}}$ values implying that the equations can be decoupled for $\boldsymbol{f_{x1}}$ non-zero [16][5]. The eigenvectors constitute the columns of the

[4] The provided mathematical procedure and final outcomes are valid even if the eigenvalues are imaginary. These eigenvalues can indeed be imaginary since they are simply mathematical entities and not physical quantities.

[5] The case when $\boldsymbol{f_{x1}}$ is zero is trivial and is solved separately; however, details are not presented here since the final results are found to be consistent with the general solution for non-zero $\boldsymbol{f_{x1}}$.

matrix $[Q]$. Using these eigenvalues and eigenvectors, the solution to Eq.(26) is simply given by [16]:

$$\begin{Bmatrix} u_y''' \\ u_y'' \\ u_z''' \\ u_z'' \end{Bmatrix} = [Q] \begin{Bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ c_3 e^{\lambda_3 x} \\ c_4 e^{\lambda_4 x} \end{Bmatrix} = \begin{bmatrix} -\boldsymbol{r} & -\boldsymbol{r} & -\dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r}} & -\dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r}} \\ -\dfrac{\boldsymbol{r}}{\boldsymbol{\lambda_1}} & -\dfrac{\boldsymbol{r}}{\boldsymbol{\lambda_2}} & -\dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{\lambda_3 r}} & -\dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{\lambda_4 r}} \\ \boldsymbol{\lambda_1} & \boldsymbol{\lambda_2} & \boldsymbol{\lambda_3} & \boldsymbol{\lambda_4} \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ c_3 e^{\lambda_3 x} \\ c_4 e^{\lambda_4 x} \end{Bmatrix} \quad (27)$$

Here, c_1, c_2, c_3 and c_4 are the constants of integration. From these four scalar equations, the general solutions for the normalized transverse displacements u_y and u_z are given by:

$$\begin{aligned} u_y &= -\frac{\boldsymbol{r}}{\boldsymbol{\lambda}_1^3} c_1 e^{\lambda_1 x} - \frac{\boldsymbol{r}}{\boldsymbol{\lambda}_2^3} c_2 e^{\lambda_2 x} - \frac{\boldsymbol{f_{x1}}}{\boldsymbol{\lambda}_3^3 \boldsymbol{r}} c_3 e^{\lambda_3 x} - \frac{\boldsymbol{f_{x1}}}{\boldsymbol{\lambda}_4^3 \boldsymbol{r}} c_4 e^{\lambda_4 x} + c_5 x + c_6 \\ u_z &= \frac{c_1 e^{\lambda_1 x}}{\boldsymbol{\lambda}_1^2} + \frac{c_2 e^{\lambda_2 x}}{\boldsymbol{\lambda}_2^2} + \frac{c_3 e^{\lambda_3 x}}{\boldsymbol{\lambda}_3^2} + \frac{c_4 e^{\lambda_4 x}}{\boldsymbol{\lambda}_4^2} + c_7 x + c_8 \end{aligned} \quad (28)$$

The constants c_5, c_6, c_7 and c_8 can be expressed in terms of c_1, c_2, c_3 and c_4 using the geometric boundary conditions at the fixed end of the beam:

$$u_y(0) = u_y'(0) = u_z(0) = u_z'(0) = 0$$

$$\begin{Bmatrix} c_5 \\ c_6 \\ c_7 \\ c_8 \end{Bmatrix} = \begin{bmatrix} \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda}_1^2} & \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda}_2^2} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda}_3^2} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda}_4^2} \\ \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda}_1^3} & \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda}_2^3} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda}_3^3} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda}_4^3} \\ -\dfrac{1}{\boldsymbol{\lambda}_1} & -\dfrac{1}{\boldsymbol{\lambda}_2} & -\dfrac{1}{\boldsymbol{\lambda}_3} & -\dfrac{1}{\boldsymbol{\lambda}_4} \\ -\dfrac{1}{\boldsymbol{\lambda}_1^2} & -\dfrac{1}{\boldsymbol{\lambda}_2^2} & -\dfrac{1}{\boldsymbol{\lambda}_3^2} & -\dfrac{1}{\boldsymbol{\lambda}_4^2} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} \quad (29)$$

The remaining four constants, c_1, c_2, c_3 and c_4, are solved using the boundary conditions at the free end of the beam

$$u_y(1) = u_{y1}, \quad u_y'(1) = \theta_{z1}, \quad u_z(1) = u_{z1}, \quad u_z'(1) = -\theta_{y1} \quad \Rightarrow$$

$\{c_1 \quad c_2 \quad c_3 \quad c_4\}^T = [C]\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T$, where

$$[C] \triangleq \begin{bmatrix} \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda_1}^3}\boldsymbol{t_{11}} & \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda_2}^3}\boldsymbol{t_{12}} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda_3}^3}\boldsymbol{t_{13}} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda_4}^3}\boldsymbol{t_{14}} \\ \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda_1}^2}\boldsymbol{t_{21}} & \dfrac{\boldsymbol{r}}{\boldsymbol{\lambda_2}^2}\boldsymbol{t_{22}} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda_3}^2}\boldsymbol{t_{23}} & \dfrac{\boldsymbol{f_{x1}}}{\boldsymbol{r\lambda_4}^2}\boldsymbol{t_{24}} \\ -\dfrac{1}{\boldsymbol{\lambda_1}^2}\boldsymbol{t_{31}} & -\dfrac{1}{\boldsymbol{\lambda_2}^2}\boldsymbol{t_{32}} & -\dfrac{1}{\boldsymbol{\lambda_3}^2}\boldsymbol{t_{33}} & -\dfrac{1}{\boldsymbol{\lambda_4}^2}\boldsymbol{t_{34}} \\ -\dfrac{1}{\boldsymbol{\lambda_1}}\boldsymbol{t_{41}} & -\dfrac{1}{\boldsymbol{\lambda_2}}\boldsymbol{t_{42}} & -\dfrac{1}{\boldsymbol{\lambda_3}}\boldsymbol{t_{43}} & -\dfrac{1}{\boldsymbol{\lambda_4}}\boldsymbol{t_{44}} \end{bmatrix}^{-1} \quad (30)$$

$$\boldsymbol{t_{1i}} = \boldsymbol{t_{3i}} \triangleq \left(1 + \lambda_i - e^{\lambda_i}\right), \qquad \boldsymbol{t_{2i}} = \boldsymbol{t_{4i}} \triangleq \left(1 - e^{\lambda_i}\right)$$

Equations (28), (29) and (30) provide the exact closed-form solutions of Eq.(26) for u_y and u_z in terms of the normalized axial coordinate x. This solution is the form of transcendental functions of the normalized axial load and moment. Substituting these solutions into Eqs.(21) and (22), followed by an integration gives the solution for u_x and θ_{xd} in terms of x.

Thus, this procedure not only captures the pertinent nonlinear effects in the beam characteristic Eqs.(21), (22) and (23) but also allows an exact closed form solution for these equations. These solutions can now be used to find a closed form strain energy expression by substituting the results of u_y, u_z, u_x and θ_{xd} in the normalized Eq.(7) which is restated here for convenience.

$$\begin{aligned} v &= \frac{1}{2}\int_0^1 \left(u_y''^2 + u_z''^2\right) dx + \frac{\boldsymbol{m_{xd1}^2}}{2k_{44}} + \frac{\boldsymbol{f_{x1}^2}}{2k_{33}} - \frac{2\boldsymbol{m_{xd1}^2 f_{x1}}}{k_{33}k_{44}^2} \\ k_{33} &\triangleq \frac{12L^2}{T_Y^2}, \quad k_{44} \triangleq \frac{GJ}{EI} \end{aligned} \quad (31)$$

The strain energy in Eq.(31) is expressed with loads in addition to displacements for ease of representation only and can be easily substituted with Eqs. (21) and (22) to express the strain energy only in terms of displacements, This expression shows the summation of the strain energies acquired from bending, torsion and elastic stretching (the first three terms of Eq. (31)) plus an extra term which come due to the combined effect of bending, torsion and elastic stretching. The final strain energy expression is given below.

$$v = \frac{1}{2}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{z1} \\ \theta_{y1} \end{Bmatrix}^T [C]^T [E][C] \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{z1} \\ \theta_{y1} \end{Bmatrix} + \frac{\boldsymbol{m_{xd1}^2}}{2k_{44}} + \frac{\boldsymbol{f_{x1}^2}}{2k_{33}} - \frac{2\boldsymbol{m_{xd1}^2 f_{x1}}}{k_{33}k_{44}^2} \quad (32)$$

$$\text{where } E_{i,j} = \left(\frac{r^2}{\lambda_i \lambda_j} + 1\right)\frac{e^{\lambda_i \lambda_j} - 1}{\lambda_i \lambda_j} \quad \text{for } i, j \in \{1,2,3,4\}$$

The first component in the energy expression above is a transcendental function in $\boldsymbol{f_{x1}}$ and $\boldsymbol{m_{xd1}}$ and can be expanded using Taylor series in terms of the $\boldsymbol{f_{x1}}$ and $\boldsymbol{m_{xd1}}$. Since the first three terms of this series up to the second power of $\boldsymbol{f_{x1}}$ and $\boldsymbol{m_{xd1}}$ contribute to 99.99% of the total energy, the expression is truncated at that point.

$$\begin{aligned} v &= \frac{1}{2}\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\} H_1 \{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T + \frac{\boldsymbol{m_{xd1}^2}}{2k_{44}} + \frac{\boldsymbol{f_{x1}^2}}{2k_{33}} \\ &+ \frac{\boldsymbol{f_{x1}^2}}{2}\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}[H_2]\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T \\ &+ \frac{\boldsymbol{m_{xd1} f_{x1}}}{2}\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}[H_3]\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T \\ &+ \frac{\boldsymbol{m_{xd1}^2}}{2}\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}[H_4]\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T \\ &- \frac{2\boldsymbol{m_{xd1}^2 f_{x1}}}{k_{33}k_{44}^2} \end{aligned} \quad (33)$$

$$H_1 \triangleq \begin{bmatrix} 12 & -6 & 0 & 0 \\ -6 & 4 & 0 & 0 \\ 0 & 0 & 12 & 6 \\ 0 & 0 & 6 & 4 \end{bmatrix}, H_2 \triangleq \begin{bmatrix} \frac{1}{700} & -\frac{1}{1400} & 0 & 0 \\ -\frac{1}{1400} & \frac{11}{6300} & 0 & 0 \\ 0 & 0 & \frac{1}{700} & \frac{1}{1400} \\ 0 & 0 & \frac{1}{1400} & \frac{11}{6300} \end{bmatrix}$$

$$H_3 \triangleq \frac{1}{60}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, H_4 \triangleq \frac{1}{20}\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

The nonlinear strain energy expression parametrically captures several fundamental effects in beam mechanics. The first three terms in Eq.(33) are the strain energy associated with linear beam mechanics. The next three terms are the energy associated with a nonlinear elastokinematic effect, also seen in nonlinear planar beam analysis [16]. As the name suggests, this effect depends both on loads and displacement, which, in this case, are axial loads and bending displacements respectively. In a finitely bent configuration, the axial force $\boldsymbol{f_{x1}}$ and the axial moment $\boldsymbol{m_{xd1}}$ cause additional curvature variation in the beam on top of bending loads $\boldsymbol{f_{y1}}$, $\boldsymbol{f_{y1}}$, $\boldsymbol{m_{y1}}$ and $\boldsymbol{m_{z1}}$. This results in a change in the end displacements u_{y1}, θ_{z1}, u_{z1}, θ_{y1}, thus producing an elastokinematic effect. Although this effect does not affect the calculation of displacements by more than 1%, its effect on the stiffness of axial stretching is as significant as linear elasticity. For torsional stiffness, this effect is one order of magnitude less than linear elasticity, which, although is not as significant as in the case of axial stretching, cannot be dropped completely.

The last energy term represents the energy due to a second nonlinear effect and is independent of transverse displacements. It is an elastic coupling between torsion and stretching. This corresponds to a small axial displacement in the presence of pure torsion and a slight change of stiffness in torsion direction due to axial force. Although this effect is at least three orders of magnitude smaller than linear elastic and kinematic effects, it becomes the only cause of axial displacement or change in stiffness in the absence of bending loads and hence is retained. In the cases of more complex mechanisms, if it can be proved that this effect cannot exist alone for any loading condition, its associated terms in energy and constraint expressions can be dropped.

The axial displacement can be derived using Eq.(21) and the bending solution given in Eq.(28). The solution is transcendental in nature and can be expanded and truncated as given in Eq.(34). The total axial displacement can be divided into four fundamental components, namely linear, kinematic, elastokinematic and elastic torsion-stretch coupling effect.

$$u_{x1} = \frac{\boldsymbol{f_{x1}}}{k_{33}} + \{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\} H_5 \{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T$$
$$+ \{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}\left[\boldsymbol{f_{x1}} H_2 + \frac{1}{2}\boldsymbol{m_{xd1}} H_3\right]\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T$$
$$- \frac{\boldsymbol{m_{xd1}^2}}{k_{44}^2 k_{33}} \tag{34}$$

where $H_5 \triangleq \begin{bmatrix} -\frac{3}{5} & \frac{1}{20} & 0 & 0 \\ \frac{1}{20} & -\frac{1}{15} & 0 & 0 \\ 0 & 0 & -\frac{3}{5} & -\frac{1}{20} \\ 0 & 0 & -\frac{1}{20} & -\frac{1}{15} \end{bmatrix}$

The axial displacement is dominated by the second term which represents the kinematic coupling between bending and axial displacements. It occurs due to the inherent arc-length preservation of beams. The stiffness, however, is dependent on the first and third terms which are due to linear and the elastokinematic effects respectively. An additional axial displacement, given by the fourth term, is caused by the twisting moment which, although is small, can be significant in the absence of axial and bending forces.

The transcendental expression for θ_{xd1} can be derived using Eq.(22) and the bending solution given in Eq.(28). The expanded and truncated form of the solution is as follows.

$$\theta_{xd1} = \frac{\boldsymbol{m_{xd1}}}{k_{44}} + \{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\} H_6 \{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T$$
$$+ \{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}\left[\boldsymbol{m_{xd1}} H_4 + \frac{1}{2}\boldsymbol{f_{x1}} H_3\right]\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\}^T$$
$$- \frac{2\boldsymbol{m_{xd1}}\boldsymbol{f_{x1}}}{k_{33}k_{44}^2} \tag{35}$$

where $H_6 \triangleq \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -1 \\ 0 & -2 & 0 & 0 \\ -2 & -1 & 0 & 0 \end{bmatrix}$

The twist in the beam is divided into four fundamental effects, similar to the axial displacement expression in Eq.(34). While the solution is dominated by the first term capturing the linear twist in the beam, the second term, representing the kinematic rotation caused bending displacements, can become significant in the absence of a twisting moment. The twist angle also has an elastokinematic component, the third term, dependent on bending displacements and axial loads. Unlike the kinematic effect, this term also contributes to the torsional stiffness. The fourth term completes this load-displacement relation showing the effect of axial forces on torsion. The effect of this term is generally very small, but can become significant for high axial forces that can occur for load bearing applications and over-constraint situations.

The strain energy model, given in Eq.(33) can be used along with constraint equations (34) and (35) in PVW to derive the bending force displacement relations. The result is given in Eq.(36) and it matches with previously published results using explicit analysis with a minor correction in the coordinate frame used for defining the twist moment [8].

$$\begin{aligned}\left\{\boldsymbol{f}_{y1} \quad \boldsymbol{m}_{z1} \quad \boldsymbol{f}_{z1} \quad \boldsymbol{m}_{y1}\right\}^T &= H_1\left\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\right\}^T \\ &+\left[2\,\boldsymbol{f}_{x1} H_5 + \boldsymbol{m}_{xd1}\left(2H_6 + H_7\right)\right]\left\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\right\}^T \\ &-\left[\boldsymbol{f}_{x1}^2 H_2 + \boldsymbol{m}_{xd1}\,\boldsymbol{f}_{x1} H_3 + \boldsymbol{m}_{xd1}^2 H_4\right]\left\{u_{y1} \quad \theta_{z1} \quad u_{z1} \quad \theta_{y1}\right\}^T\end{aligned} \tag{36}$$

Where

$$H_7 \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

As expected the linear (first term), nonlinear kinematic (second term) and nonlinear elastokinematic (third term) behaviors have been captured in bending direction load-displacement relation in Eq.(36).

5. DISCUSSION AND CONCLUSION

A nonlinear closed form strain energy model for symmetric spatial beam flexure, compatible with the previously presented nonlinear beam constraint model, is formulated. In addition to linear effects, the model explicitly shows the possible changes in energy and constraint relations via closed-form parametric expressions that arise due to the nonlinear interactions of the bending, axial stretching and torsion directions. The formulation also shows an interesting similarity between the twist angle expression in Eq.(35) and axial translation expression in Eq.(34), which shows the constraint-like behavior of θ_{xd} in the application of energy methods, contrary to its expected DoF–like behavior due to its relatively low stiffness.

This analysis paves the way for analyzing complex three-dimensional flexure mechanisms, comprising several spatial beam flexures, using energy methods. Since no assumption, other than symmetry, is made for the beam cross-section, this formulation is applicable to beams with circular, square and other regular polygon shaped cross-sections. Furthermore, the simplicity of the nonlinear strain energy is expected to help in the analysis of complicated flexure mechanisms. Future work includes analyzing complex parallel–kinematic and serial spatial flexure mechanisms using this methodology followed by an FEA and experimental validation.

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REFERENCES

1. Jones, R.V., 1988, *Instruments and Experiences: Papers on Measurement and Instrument Design*, John Wiley & Sons, New York, NY
2. Smith, S.T., 2000, *Flexures: Elements of Elastic Mechanisms*, Gordon and Breach Science Publishers, New York, NY
3. Blanding, D. L., 1999, *Exact Constraint: Machine Design Using Kinematic Principles*, ASME Press, New York
4. Su, H.-J., and Tari, H., 2010, "Realizing orthogonal motions with wire flexures connected in parallel", *ASME Journal of Mechanical Design*, 132(12)
5. Awtar, S., Slocum, A.H., and Sevincer, E., 2006, "Characteristics of Beam-based Flexure Modules", *ASME Journal of Mechanical Design*, 129 (6), pp 625-639
6. Awtar, S. and Slocum, A.H., 2007, "Constraint-based Design of Parallel Kinematic XY Flexure Mechanisms", *ASME Journal of Mechanical Design*, 129 (8), pp. 816-830
7. Zelenika, S., and DeBona, F., 2002, "Analytical and Experimental Characterization of High Precision Flexural Pivots subjected to Lateral Loads", *Precision Engineering*, 26, pp. 381-388
8. Sen S. and Awtar S., 2010, "Nonlinear Constraint Model For Symmetric Three-Dimensional Beams", *Proc. ASME IDETC & CIE 2010*, Montreal, Canada, Paper # 28953
9. Tauchert, T. R., 1974, *Energy Principles in Structural Mechanics*, McGraw-Hill, New York
10. Timoshenko S. and Goodier J. N., 1969, *Theory of Elasticity*, McGraw-Hill, New York
11. Crandall, S. H., Dahl, N. C., and Lardner T. J., 1972, *An Introduction to the Mechanics of Solids*, McGraw-Hill Book Company, NY
12. Crespo DaSilva, M. R. M, 1988, "Non-Linear Flexural-flexural-torsional-Extensional Dynamics of Beams-I. Formulation", *Int. J. of Solid Structures*, v 24, n12, pp 1225-1234
13. Hodges, D. H. and Dowell, E. H., 1974, "Nonlinear equations of motion for the elastic bending and torsion of twisted non-uniform rotor blades", *NASA Technical Note D-7818*
14. Annigeri, B. S., Cassenti, B. N. and Dennis, A. J., 1985, "Kinematics of small and large deformations of continua", *Engineering Computations*, 2, pp 247-256
15. Mayo, J., and Dominguez, J., 1992, "Geometrically Nonlinear Coupling between Axial and Flexural Modes of Deformation of Multibody Systems*", Dynamics of Flexible Multibody Systems: Theory and Experiments*, Winter Annual Meeting of ASME, Anaheim CA, AMD-v.141, DSC-v.37, pp 95-103
16. Chen, C., 1999, *Linear Sytem Theory and Design,* Oxford University Press, NY
17. Awtar, S., and Sen, S., 2010, "A Generalized Constraint Model for Two Dimensional Beam Flexures: Non-Linear Load-Displacement Formulation", *ASME Journal of Mechanical Design*, 132 (8)