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BEAM CONSTRAINT MODEL: A NON-LINEAR STRAIN ENERGY FORMULATION FOR GENERALIZED TWO-DIMENSIONAL BEAM FLEXURES

Shorya Awtar* and Shiladitya Sen
Precision Systems Design Laboratory
Mechanical Engineering, University of Michigan
Ann Arbor MI 48109

ABSTRACT

In the past, we have introduced the Beam Constraint Model (BCM), which captures pertinent non-linearities to predict the constraint characteristics of a generalized beam flexure in terms of its stiffness and error motions. In this paper, a non-linear strain energy formulation for the beam flexure, consistent with the transverse-direction load-displacement and axial-direction geometric constraint relations in the BCM, is presented. An explicit strain energy expression, in terms of beam end-displacements, that accommodates generalized loading conditions, boundary conditions, initial curvature, and beam shape is derived. Using the Principle of Virtual Work, this strain energy expression for a generalized beam is employed in determining the load-displacement relations, and therefore constraint characteristics, for flexure mechanisms comprising multiple beams. The benefit of this approach is evident in its mathematical efficiency and succinctness, which is to be expected with the use of energy methods. All analytical results are validated to a high degree of accuracy via non-linear Finite Element Analysis. Furthermore, the proposed energy formulation leads to new insights into the nature of the BCM.

1. INTRODUCTION AND BACKGROUND

The Beam Constraint Model (BCM) is a closed-form, parametric, and generalized model that captures the constraint characteristics of a beam flexure in terms of its stiffness and error motions. While the background and motivation for this model are presented in prior publications [1], a brief review is provided in the following paragraphs.

Fig.1 illustrates a simple beam (initially straight, uniform cross-section) of length L, thickness T, and depth H, interconnecting rigid bodies (1) and (2). The beam is subject to generalized end-loads $\boldsymbol{F_{XL}}$, $\boldsymbol{F_{YL}}$, and $\boldsymbol{M_{ZL}}$, which result in end-displacements U_{XL}, U_{YL}, and θ_{ZL} with respect to the coordinate frame X-Y-Z. Transverse direction displacements U_{YL} and θ_{ZL} can be recognized to be the Degrees of Freedom (DoF) of this flexure unit. The axial direction U_{XL} displacement represents a Degree of Constraint (DoC).

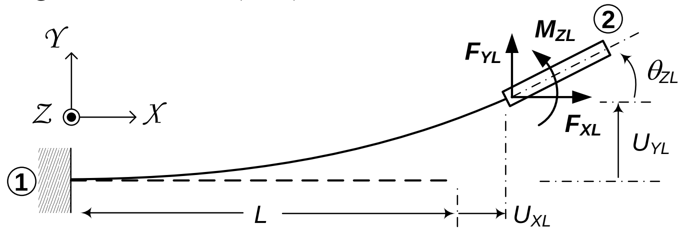

Fig. 1 Simple Beam Flexure

The BCM, expressed in terms of the transverse-direction end load-displacement relation and the axial direction geometric constraint relation for the above beam flexure, has been derived previously [1]. This model captures the non-linearities associated with applying load equilibrium in the deformed state, but neglects those associated with beam curvature. It has been shown that the former are crucial in accurately capturing the constraint characteristics of a beam flexure in terms of its stiffness and error motions.

The BCM has also been extended to include beams with arbitrary end-loads, initial and boundary conditions, beam shape, and temperature distribution. Furthermore, it has been employed to accurately determine the load-displacement relations, and therefore constraint characteristics, of more complex flexure mechanisms that comprise beam flexures. However, the direct application of the BCM for this purpose proves to be mathematically tedious since all the internal loads and displacements associated with each beam have to be taken into account.

This limitation provides the motivation for the energy-based formulation of the BCM presented in this paper. In particular, the Principle of Virtual Work (PVW) is employed

*Corresponding Author (awtar@umich.edu, 734-615-0285)

because it eliminates the need to consider internal loads and load equilibrium for each constituent beam in a flexure mechanism [2]. The first requirement for applying PVW on an elastic system is the determination of the strain energy corresponding to an arbitrary deformed state. This is non-trivial because we want to capture a certain class of non-linearities in the BCM, while neglecting others. Therefore, the assumptions in the strain energy formulation have to be consistent with those made in the direct determination of the transverse and axial relations in the BCM. This is addressed in Section 2 of this paper, where non-linear strain and strain energy expressions are developed.

In Section 3, we derive expressions for the transverse direction load-displacement relation, axial direction geometric constraint relation, and total strain energy explicitly in terms of the end-loads and end-displacements, for a variable cross-section beam. Upon simplification (series expansion and truncation), these three expressions reveal beam characteristic coefficients, which may be classified as stiffness, constraint, and energy coefficients, respectively.

In Section 4, we make use of two separate energy based arguments to establish fundamental relations between the beam characteristic coefficients. The first is based on the PVW and the second is based on the conservation of energy. The application of PVW at this stage also provides a consistent truncation scheme for the infinite series in the transverse load-displacement, axial geometric constraint, and strain energy expressions, as shown in Section 5. Since both the axial constraint and strain energy expressions exhibit a dependence on the axial load, the two expressions are combined to yield a strain energy expression free of axial load terms. The resulting expression represents the energy formulation of the BCM. This strain energy expression for a single beam is now in a form that may be employed in conducting the load-displacement analysis of a multi-beam flexure mechanism using energy methods. This is illustrated in Section 6, where the effectiveness and utility of the BCM energy formulation is highlighted using the PVW. In Section 7, the BCM energy formulation is further generalized to incorporate an initially slanted and/or curved beam.

2. NON-LINEAR STRAIN AND STRAIN ENERGY FORMULATION FOR THE BEAM FLEXURE

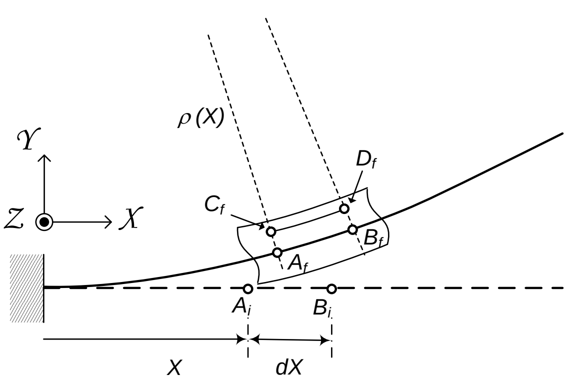

Fig.2 Undeformed and Deformed Beam Geometries

Fig.2 illustrates the neutral axis of a simple beam (initially straight and uniform-thickness) in its undeformed (dashed line) and deformed (solid line) geometries, with respect to the indicated X-Y-Z coordinate axes. $U_X(X)$ and $U_Y(X)$ represent the X and Y direction displacements of any point A_i on the beam's neutral axis. An element A_iB_i along the undeformed beam neutral axis assumes a new position and orientation A_fB_f after deformation. Therefore, the axial strain (ε_{xx}) at location X along the neutral axis can be stated as:

$$\begin{aligned}\varepsilon_{xx}(X,0) &= \frac{A_fB_f - A_iB_i}{A_iB_i} = \left[\left(1+U'_X\right)^2 + U'^2_Y\right]^{1/2} - 1 \\ &= U'_X\left(1-\frac{7}{16}U'^2_X - \frac{1}{2}U'^2_Y + \ldots\right) + \frac{U'^2_Y}{2}\left(1-\frac{5}{16}U'^2_X - \frac{1}{4}U'^2_Y + \ldots\right) \\ &\approx U'_X + \frac{U'^2_Y}{2} \qquad (1)\end{aligned}$$

It is physically obvious that the axial direction U'_X is much less than the transverse direction U'_Y. For U'_Y less than 0.1, second and higher power terms in U'_X and U'_Y may be dropped with respect to 1 in the infinite series expansions above, with less than 1% error. Note, however, that in the final form of the strain expression, the second-power U'^2_Y term has been retained with respect to the first-power U'_X – a key aspect of the non-linear strain formulation. This second-power U'^2_Y term appears because the deformed geometry (translation and rotation) of the beam neutral axis has been considered in the strain formulation, and it is comparable to the first-power U'_X even for small displacements. This expression for strain represents the true stretch of an element along the neutral axis and inherently captures the kinematics associated with the geometric constraint in the problem, i.e., beam arc length conservation.

Next, the axial strain of an element at distance Y from the neutral axis, along the Y direction, may be determined by calculating the additional length change of the element C_fD_f with respect to element A_fB_f (Fig.2). Assuming that plane sections remain plane and normal to the neutral axis after deformation (Bernoulli's assumptions), Eq.(1) may be augmented to show that:

$$\begin{aligned}\varepsilon_{xx}(X,Y) &\approx U'_X + \frac{U'^2_Y}{2} - \frac{Y}{\rho(X)} = U'_X + \frac{U'^2_Y}{2} - \frac{U''_Y}{\left(1-U'^2_Y\right)^{1/2}}Y \\ &\approx U'_X + \frac{U'^2_Y}{2} - U''_Y\,Y \qquad (2)\end{aligned}$$

where $\rho(X)$ is the radius of curvature at a beam neutral axis location that was originally at X before deformation [3]. Consistent with the previous approximations, in the final step above, the second and higher power terms in U'_Y have been neglected with respect to 1 in the curvature expression.

Next, the net strain energy in the beam may be determined to be the following:

$$V = \iiint_{Volume} \frac{E}{2}\varepsilon^2_{XX}\, dA dX \qquad (3)$$

where E represents the Young's modulus of the material for an XY plane-stress condition and the plate modulus for an XY plane-strain condition. Substituting Eq.(2) in Eq.(3), followed by some mathematical steps, yields

$$V = \frac{EA}{2}\int_0^L \left\{ U'_X + \frac{1}{2}U'^2_Y \right\}^2 dX \;+\; \frac{EI_{ZZ}}{2}\int_0^L U''^2_Y dX \tag{4}$$

where A and I_{zz} are the area and second moment of area, respectively. The key difference in this strain energy expression, compared to a linear formulation, is the presence of the second-power U'^2_Y term in the first integral above, which inherently captures the geometric constraint of beam arc length conservation. Expression (4) is also in agreement with previous non-linear strain energy formulations [4].

With the strain energy thus determined and the geometric boundary conditions known at the beam root ($U_X(0)=0$, $U_Y(0)=0$, and $U'_Y(0)=0$), the Principle of Virtual Work (PVW) may be applied to the beam flexure (Fig.1) to yield the following beam governing equations and natural boundary conditions.

Governing Equations:

$$EI_{ZZ}U_Y^{iv} - \boldsymbol{F_{XL}}U''_Y = 0 \tag{5}$$

$$U'_X + \frac{1}{2}U'^2_Y = \frac{\boldsymbol{F_{XL}}}{EA} \tag{6}$$

Natural Boundary Conditions:

$$-EI_{ZZ}U'''_Y(L) + \boldsymbol{F_{XL}}U'_Y(L) = \boldsymbol{F_{YL}} \tag{7}$$

$$EI_{ZZ}U''_Y(L) - \boldsymbol{M_{ZL}} = 0 \tag{8}$$

Eq.(5) provides the recognizable transverse direction beam governing equation. This fourth order linear differential equation in U_Y is exactly the same as the one obtained in the direct formulation [1], which is to be expected because the set of assumptions made in both cases are identical. Eq.(6) provides the axial direction geometric constraint equation for the beam flexure. This reaffirms that the geometric constraint associated with the beam arc length is inherently captured in the above strain and strain energy formulations (Eqs. (2) and (4)) and therefore is not needed explicitly in the application of the PVW; instead, the geometric constraint relation falls out of the PVW application. Eq.(6) may be integrated once to yield the following axial direction relation, same as the result from the direct formulation [1]:

$$U_{XL} = \frac{\boldsymbol{F_{XL}}L}{EA} - \frac{1}{2}\int_0^L U'^2_Y dX = U^{(e)}_{XL} - \frac{1}{2}\int_0^L U'^2_Y dX \tag{9}$$

The first term above represents the linear elastic stretching of the beam in the axial direction in response to an axial force $\boldsymbol{F_{XL}}$ and is denoted by $U^{(e)}_{XL}$. The second term captures the geometric constraint of beam arc length conservation. Eq.(6) also corroborates the fact that the Left-Hand Side represents the true axial strain in the beam due to stretching, which remains constant throughout the beam length since the axial load and therefore stress, given by the Right-Hand Side of this equation, remains constant.

For subsequent application of the PVW, the strain energy expression of Eq.(4) may be further simplified by employing Eq.(6) and invoking the definition of $U^{(e)}_{XL}$, to yield

$$V = \frac{EI_{ZZ}}{2}\int_0^L U''^2_Y dX \;+\; \frac{AE}{2L}\left(U^{(e)}_{XL}\right)^2 \tag{10}$$

This clearly identifies the separate contributions to the non-linear strain energy from beam bending and beam axial stretching. It will be seen later that the first term not only includes the bending deformation induced by the transverse loads but also that induced by the axial load.

Having thus established the consistency of the non-linear strain energy expressions (4) and (10) with the previously reported transverse-direction beam governing equation, axial geometric constraint relation, and associated boundary conditions that led to the BCM [1], we now proceed to use these strain energy expressions as the basis for an energy-based BCM formulation.

At this stage, a normalization scheme is introduced to simplify mathematical expressions and their manipulation in the rest of this paper. All loads, displacements, position coordinates, stiffness, energy, and work terms are normalized with respect to the beam geometry and material parameters: displacements, lengths, and coordinates are normalized by the beam length L; forces by EI_{zz}/L^2; and moments, work, and strain energy by EI_{zz}/L:

$$\boldsymbol{f_{x1}} \triangleq \frac{\boldsymbol{F_{XL}}L^2}{EI_{ZZ}} \;;\; \boldsymbol{f_{y1}} \triangleq \frac{\boldsymbol{F_{YL}}L^2}{EI_{ZZ}} \;;$$

$$\boldsymbol{m_{z1}} \triangleq \frac{\boldsymbol{M_{ZL}}L}{EI_{ZZ}} \;;\; v \triangleq \frac{VL}{EI_{ZZ}} \;;\; w \triangleq \frac{WL}{EI_{ZZ}}$$

$$x \triangleq \frac{X}{L} \;;\; u_x(x) \triangleq \frac{U_X(X)}{L} \;;\; u_y(x) \triangleq \frac{U_Y(X)}{L} \;;$$

$$u_{x1} \triangleq \frac{U_{XL}}{L} \;;\; u_{y1} \triangleq \frac{U_{YL}}{L} \;;\; \theta_{z1} \triangleq \theta_{ZL} = U'_Y(X)$$

3. BEAM RELATIONS IN TERMS OF END LOADS AND DISPLACEMENTS

For a simple beam, the beam governing equation (Eq.(5)) and associated geometric boundary conditions may be solved in closed-form,

$$\begin{aligned}
&u_y(x) = c_1 e^{rx} + c_2 e^{-rx} + c_3 x + c_4 \quad , \quad \text{where } \boldsymbol{r}^2 \triangleq \boldsymbol{f_{x1}} \\
&c_1 = \frac{\boldsymbol{r}(e^{-r}-1)u_{y1} + (e^{-r}+\boldsymbol{r}-1)\theta_{z1}}{\boldsymbol{r}\left[\boldsymbol{r}(e^{r}-e^{-r}) - 2(e^{r}+e^{-r}) + 4\right]}; \\
&c_2 = \frac{\boldsymbol{r}(e^{r}-1)u_{y1} - (e^{r}-\boldsymbol{r}-1)\theta_{z1}}{\boldsymbol{r}\left[\boldsymbol{r}(e^{r}-e^{-r}) - 2(e^{r}+e^{-r}) + 4\right]} \\
&c_3 = \boldsymbol{r}(c_2 - c_1) \;\;;\;\; c_4 = -c_1 - c_2
\end{aligned} \tag{11}$$

The intermediary variable $\boldsymbol{r}$ ($\triangleq \boldsymbol{f_{x1}}^{1/2}$) is introduced temporarily for mathematical convenience. The application of natural boundary conditions (7)-(8) in the above expression yields the following transverse-direction end load-displacement relations:

$$\begin{Bmatrix} \boldsymbol{f}_{y1} \\ \boldsymbol{m}_{z1} \end{Bmatrix} = [k] \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix}$$

$$[k] \triangleq \begin{bmatrix} \dfrac{\boldsymbol{r}^3 \sinh \boldsymbol{r}}{\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2} & -\dfrac{\boldsymbol{r}^2 \left(\cosh \boldsymbol{r} - 1\right)}{\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2} \\ -\dfrac{\boldsymbol{r}^2 \left(\cosh \boldsymbol{r} - 1\right)}{\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2} & \dfrac{\boldsymbol{r}^2 \cosh \boldsymbol{r} - \boldsymbol{r}\sinh \boldsymbol{r}}{\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2} \end{bmatrix} \tag{12}$$

In this non-linear formulation, the stiffness terms are no longer simply elastic terms as in the purely linear case, but instead are functions of the axial load $\boldsymbol{f}_{x1}$. These transcendental expressions may be expanded in $\boldsymbol{f}_{x1}$ to yield the following infinite series:

$$\begin{aligned} \begin{Bmatrix} \boldsymbol{f}_{y1} \\ \boldsymbol{m}_{z1} \end{Bmatrix} = & \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ \theta_{z1} \end{Bmatrix} + \boldsymbol{f}_{x1} \begin{bmatrix} \frac{6}{5} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2}{15} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + \boldsymbol{f}_{x1}^2 \begin{bmatrix} -\frac{1}{700} & \frac{1}{1400} \\ \frac{1}{1400} & -\frac{11}{6300} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + \boldsymbol{f}_{x1}^3 \begin{bmatrix} \frac{1}{63000} & -\frac{1}{126000} \\ -\frac{1}{126000} & \frac{1}{27000} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \ldots \end{aligned} \tag{13}$$

In the BCM, the first matrix in the above series captures the elastic stiffness and the second matrix captures load-stiffening, which quantifies the change in DoF direction effective stiffness in the presence of a DoC load [5-6]. Terms associated with higher powers of $\boldsymbol{f}_{x1}$ are found to have negligible contributions for practical load and displacement ranges of interest.

Next, the solution given by Eq.(11) may be substituted in Eq.(9) to obtain the axial-direction geometric constraint equation in terms of end-loads and displacements:

$$u_{x1} = u_{x1}^{(e)} + \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \tag{14}$$

where $u_{x1}^{(e)} = \dfrac{\boldsymbol{f}_{x1}}{k_{33}}$ and $k_{33} = \dfrac{12}{t^2}$

$$\begin{aligned} g_{11} &= -\frac{\boldsymbol{r}^2 \left(\cosh \boldsymbol{r} + 2\right)\left(\cosh \boldsymbol{r} - 1\right) - 3\boldsymbol{r}\sinh \boldsymbol{r}\left(\cosh \boldsymbol{r} - 1\right)}{2\left(\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2\right)^2} \\ g_{12} &= \frac{\boldsymbol{r}^2 \left(\cosh \boldsymbol{r} - 1\right) + \boldsymbol{r}\sinh \boldsymbol{r}\left(\cosh \boldsymbol{r} - 1\right) - 4\left(\cosh \boldsymbol{r} - 1\right)^2}{4\left(\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2\right)^2} \\ g_{22} &= \frac{-\boldsymbol{r}^2 \sinh \boldsymbol{r}\left(\cosh \boldsymbol{r} + 2\right) + 2\boldsymbol{r}\left(2\cosh \boldsymbol{r} + 1\right)\left(\cosh \boldsymbol{r} - 1\right)}{4\boldsymbol{r}\left(\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2\right)^2} \\ &\quad + \frac{\boldsymbol{r}^3 - 2\sinh \boldsymbol{r}\left(\cosh \boldsymbol{r} - 1\right)}{4\boldsymbol{r}\left(\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2\right)^2} \end{aligned} \tag{15}$$

This axial direction relation for the DoC end-displacement u_{x1} in terms of DoF end-displacements, u_{y1} and θ_{z1}, and DoC end-load $\boldsymbol{f}_{x1}$ is as expected [5-6]. Since this expression arises from the purely geometric constraint of constant beam arc length, the presence of the axial load $\boldsymbol{f}_{x1}$ in the constraint terms g's is somewhat surprising. While uncommon in mechanics, this does highlight the unique attributes of distributed compliance mechanisms, and will be shown to be responsible for the elastokinematic effect in the BCM. The transcendental expressions for the constraint terms may be expanded in terms of $\boldsymbol{f}_{x1}$ to yield the following infinite series:

$$\begin{aligned} u_{x1} = u_{x1}^{(e)} & + \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{1}{20} \\ \frac{1}{20} & -\frac{1}{15} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + \boldsymbol{f}_{x1} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} \frac{1}{700} & -\frac{1}{1400} \\ -\frac{1}{1400} & \frac{11}{6300} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + \boldsymbol{f}_{x1}^2 \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} -\frac{1}{42000} & \frac{1}{84000} \\ \frac{1}{84000} & -\frac{1}{18000} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \ldots \end{aligned} \tag{16}$$

The first term in this series expansion (zeroth power of $\boldsymbol{f}_{x1}$) indicates a component that is explicitly and exclusively dependent on the transverse end-displacements u_{y1} and θ_{z1}, and is independent of any loads. Therefore, this term is referred to as the kinematic component of the axial displacement and denoted by $u_{x1}^{(k)}$. The next term, although small compared to the first term, is comparable to the purely elastic term $u_{x1}^{(e)}$, and therefore cannot be ignored. Even though this term arises from the geometric constraint of beam arc-length conservation, it does have a linear dependence on $\boldsymbol{f}_{x1}$, and therefore contributes to the compliance along the DoC direction. This term, referred to as the elastokinematic component in the BCM and denoted by $u_{x1}^{(e-k)}$, is unusual and a unique outcome of distributed compliance. The consideration of the beam in its deformed configuration in formulating the non-linear strain and strain energy in Section 3, ensures that the contribution of the axial load $\boldsymbol{f}_{x1}$ to the bending moments at any given beam cross-section is appropriately captured. Because of the beam's distributed compliance, this additional bending moment causes a change in its deformation, which produces the elastokinematic displacement $u_{x1}^{(e-k)}$ along the DoC direction, even as the DoF displacements u_{y1} and θ_{z1} remain constant. Second and higher power $\boldsymbol{f}_{x1}$ terms in the above expression have a negligible contribution in the load and displacement ranges of interest.

Next, we proceed to determine the strain energy in terms of end-displacements. Note that it would be incorrect to simply employ the stiffness expression (12) determined above to find the strain energy. The stiffness given by this expression is the effective stiffness in the sense that it also captures load-stiffening, which is a consequence of the geometry and not deformation and therefore does not contribute to the strain energy. The strain energy may be accurately determined by substituting the beam deformation expression (11) in Eq.(10).

$$v=\frac{1}{2}\left\{u_{y1}\quad \theta_{z1}\right\}\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\end{Bmatrix}+\frac{1}{2}k_{33}\left(u_{x1}^{(e)}\right)^2$$

where $$v_{11}=\frac{\boldsymbol{r}^3\left(\sinh\boldsymbol{r}-\boldsymbol{r}\right)\left(\cosh\boldsymbol{r}-1\right)}{\left(\boldsymbol{r}\sinh\boldsymbol{r}-2\cosh\boldsymbol{r}+2\right)^2}$$

$$v_{12}=v_{21}=-\frac{1}{2}\frac{\boldsymbol{r}^3\left(\sinh\boldsymbol{r}-\boldsymbol{r}\right)\left(\cosh\boldsymbol{r}-1\right)}{\left(\boldsymbol{r}\sinh\boldsymbol{r}-2\cosh\boldsymbol{r}+2\right)^2} \tag{17}$$

$$v_{22}=\frac{1}{2}\frac{\left(\boldsymbol{r}^3\sinh\boldsymbol{r}-2\boldsymbol{r}^2\cosh\boldsymbol{r}+2\boldsymbol{r}\sinh\boldsymbol{r}\right)\left(\cosh\boldsymbol{r}-1\right)}{\left(\boldsymbol{r}\sinh\boldsymbol{r}-2\cosh\boldsymbol{r}+2\right)^2}-\frac{1}{2}\frac{\boldsymbol{r}^3\left(\sinh\boldsymbol{r}-\boldsymbol{r}\right)}{\left(\boldsymbol{r}\sinh\boldsymbol{r}-2\cosh\boldsymbol{r}+2\right)^2}$$

The first term above represents energy due to beam bending, while the second term represents energy due to the axial stretching of the beam arc-length. What is unusual is that the bending strain energy is not simply dependent on the transverse end-displacements u_{y1} and θ_{z1}, but also on the axial load ($\boldsymbol{f}_{x1}\triangleq\boldsymbol{r}^2$). This is simply a consequence of the fact that even when the transverse end-displacements are held fixed, the axial load can produce additional bending moment along the beam shape which results in an additional bending deformation of the beam, thus contributing an additional component of energy. This is the manifestation of the elastokinematic effect in the strain energy domain. The transcendental terms in Eq.(17) may be expanded in $\boldsymbol{f}_{x1}$ to yield the following infinite series:

$$\begin{aligned}v=&\frac{1}{2}k_{33}\left(u_{x1}^{(e)}\right)^2+\frac{1}{2}\left\{u_{y1}\quad \theta_{z1}\right\}\begin{bmatrix}12 & -6\\ -6 & 4\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\end{Bmatrix}\\ &+\frac{1}{2}\boldsymbol{f}_{x1}^{2}\left\{u_{y1}\quad \theta_{z1}\right\}\begin{bmatrix}\frac{1}{700} & -\frac{1}{1400}\\ -\frac{1}{1400} & \frac{11}{6300}\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\end{Bmatrix}\\ &+\frac{1}{2}\boldsymbol{f}_{x1}^{3}\left\{u_{y1}\quad \theta_{z1}\right\}\begin{bmatrix}-\frac{1}{31500} & \frac{1}{63000}\\ \frac{1}{63000} & -\frac{1}{13500}\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\end{Bmatrix}+\dots\end{aligned} \tag{18}$$

It may be shown that, for practical axial load and transverse displacement ranges, only the first load term, which is quadratic in $\boldsymbol{f}_{x1}$, is the most significant and the higher power terms may be neglected. Moreover, there appears to be some similarity between the stiffness and energy coefficients of the corresponding powers of $\boldsymbol{f}_{x1}$ in Eqs. (13) and (18). In fact, the zeroth-power coefficients are the same, showing that the biggest portion of the strain energy comes from the elastic stiffness in the transverse direction. Interestingly, there is no first-power term in the energy expression corresponding to the first-power term in the stiffness expression, which is associated with the load-stiffening effect. This agrees with a physical understanding of the system – since the load-stiffening effect is a consequence of geometry and not deformation, it should not contribute any strain energy. Subsequent stiffness and energy coefficients of corresponding powers of $\boldsymbol{f}_{x1}$ show some similarity but are not identical. Furthermore, there also appears to be some similarity between the stiffness coefficients of a certain power of $\boldsymbol{f}_{x1}$ in Eq.(16) and the constraint coefficients associated with one lower power of $\boldsymbol{f}_{x1}$ in Eq.(18). Thus, the natural question that arises is whether there is some underlying relationship between these stiffness, constraint, and energy coefficients of the various powers of $\boldsymbol{f}_{x1}$ in Eqs.(13), (16), and(18), respectively, or if this similarity is merely a coincidence.

Another question that remains to be answered is where to truncate the infinite series associated with the stiffness, constraint, and energy expressions for the purpose of obtaining an accurate yet compact Beam Constraint Model. In the explicit formulation, we dealt with only two relations – the transverse direction load-displacement relation and the axial direction geometric constraint relation. Both relations were truncated to keep only the first-power term in $\boldsymbol{f}_{x1}$, and it was noted that this led to errors less than *3%* over an axial load range of ±*5* and transverse displacement range of ±*0.1* [1, 5-6]. Now we have a third relation that captures the strain energy of the beam flexure. It is not clear if this third expression, or even the first two, can be truncated independent of each other, or if a certain scheme has to be followed so that the truncated expressions are 'consistent' with respect to each other. The similarity between the coefficients noted earlier, seems to indicate that there should be a consistent truncation scheme that at least ensures that the PVW is valid even for the truncated expressions.

However, before addressing the above two questions in Section 4, we first proceed to show that the format of Eqs.(13), (16), and(18) accommodates any general beam shape and not just a uniform-thickness beam. The beam deformation, end-loading, and end-displacements representation for the variable cross-section beam remains the same as in Fig.1. The modeling assumptions are also the same as earlier, except that the beam thickness is now a function of X: $T\left(X\right)=T_0\xi\left(X\right)$, where T_0 is the nominal beam thickness at the beam root and $\xi(X)$ represents the beam shape variation. Thus, the second moment of area becomes $I_{ZZ}\left(X\right)=I_{ZZ0}\xi^3\left(X\right)$. The normalization scheme remains the same as earlier, with the exception that I_{ZZ0} is now used in place of I_{ZZ}. Employing the PVW, one may derive the following normalized governing equations and natural boundary conditions for this case:

<u>Governing Equations:</u>

$$\left(\xi^3(x)\,u_y''(x)\right)''-\boldsymbol{f}_{x1}\,u_y''(x)=0 \tag{19}$$

$$u_x'(x)+\frac{1}{2}u_y'^2(x)=\boldsymbol{f}_{x1}\,\frac{t_0^2}{12\xi(x)} \tag{20}$$

<u>Natural Boundary Conditions:</u>

$$-\left\{\xi^3(1)\,u_y''(1)\right\}'+\boldsymbol{f}_{x1}\,u_y'(1)=\boldsymbol{f}_{y1} \tag{21}$$

$$\xi^3(1)\, u_y''(1) - \boldsymbol{m}_{z1} = 0 \tag{22}$$

Given the arbitrariness of $\xi(x)$, a closed-form solution to this ordinary differential equation with variable coefficients (Eq.(19)) is no longer possible. Nevertheless, the equation and boundary conditions remain linear in the transverse loads ($\boldsymbol{f}_{y1}$ and $\boldsymbol{m}_{z1}$) and transverse displacements ($u_y(x)$ and its derivatives). This implies that the resulting relation between the transverse end-loads and displacement also has to be linear, of the form:

$$\begin{Bmatrix} \boldsymbol{f}_{y1} \\ \boldsymbol{m}_{z1} \end{Bmatrix} = \begin{bmatrix} k_{11}\left(\boldsymbol{f}_{x1};\xi(x)\right) & k_{12}\left(\boldsymbol{f}_{x1};\xi(x)\right) \\ k_{21}\left(\boldsymbol{f}_{x1};\xi(x)\right) & k_{22}\left(\boldsymbol{f}_{x1};\xi(x)\right) \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \tag{23}$$

The effective stiffness terms (k's) will now be some functions of the axial load $\boldsymbol{f}_{x1}$, dictated by the beam shape $\xi(x)$, and might be difficult or impossible to determine in closed-form. Nevertheless, these functions may certainly be expanded as a generic infinite series in $\boldsymbol{f}_{x1}$,

$$\begin{aligned} \begin{Bmatrix} \boldsymbol{f}_{y1} \\ \boldsymbol{m}_{z1} \end{Bmatrix} &= \begin{bmatrix} k_{11}^{(0)} & k_{12}^{(0)} \\ k_{12}^{(0)} & k_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \boldsymbol{f}_{x1} \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{12}^{(1)} & k_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ &\quad + \boldsymbol{f}_{x1}^{\,2} \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{12}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \ldots \\ &= \sum_{n=0}^{\infty} \boldsymbol{f}_{x1}^{\,n} \begin{bmatrix} k_{11}^{(n)} & k_{12}^{(n)} \\ k_{12}^{(n)} & k_{22}^{(n)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \end{aligned} \tag{24}$$

Similarly, it may be shown that irrespective of the beam shape, the constraint equation may be stated and expanded as:

$$\begin{aligned} u_{x1} &= u_{x1}^{(e)} + \left\{ u_{y1} \quad \theta_{z1} \right\} \begin{bmatrix} g_{11}\left(\boldsymbol{f}_{x1};\xi(x)\right) & g_{12}\left(\boldsymbol{f}_{x1};\xi(x)\right) \\ g_{21}\left(\boldsymbol{f}_{x1};\xi(x)\right) & g_{22}\left(\boldsymbol{f}_{x1};\xi(x)\right) \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ &= \frac{\boldsymbol{f}_{x1}}{k_{33}} + \sum_{n=0}^{\infty} \boldsymbol{f}_{x1}^{\,n} \left\{ u_{y1} \quad \theta_{z1} \right\} \begin{bmatrix} g_{11}^{(n)} & g_{12}^{(n)} \\ g_{12}^{(n)} & g_{22}^{(n)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \end{aligned} \tag{25}$$

Along the same lines, the strain energy for a variable cross-section beam may be shown to be quadratic in the transverse displacements, u_{y1} and θ_{z1}, and some unknown function of the axial load $\boldsymbol{f}_{x1}$. This expression may be expanded as follows:

$$\begin{aligned} v &= \frac{1}{2}\left\{ u_{y1} \quad \theta_{z1} \right\} \begin{bmatrix} v_{11}\left(\boldsymbol{f}_{x1};\xi(x)\right) & v_{12}\left(\boldsymbol{f}_{x1};\xi(x)\right) \\ v_{21}\left(\boldsymbol{f}_{x1};\xi(x)\right) & v_{22}\left(\boldsymbol{f}_{x1};\xi(x)\right) \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \frac{1}{2} k_{33}\, u_{x1}^{(e)^2} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \boldsymbol{f}_{x1}^{\,n} \left\{ u_{y1} \quad \theta_{z1} \right\} \begin{bmatrix} v_{11}^{(n)} & v_{12}^{(n)} \\ v_{12}^{(n)} & v_{22}^{(n)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \frac{1}{2} k_{33}\, u_{x1}^{(e)^2} \end{aligned} \tag{26}$$

4. FUNDAMENTAL RELATIONS BETWEEN BEAM CHARACTERISTIC COEFFICIENTS

Having derived the generic expressions (24), (25), and (26) for the transverse stiffness, axial constraint, and strain energy for an initially straight beam with any generalized shape, the next step is to determine if there are any fundamental relations between these three expressions, and their associated beam characteristic coefficients. To do so, we employ the PVW once again. These three expressions, explicit in terms of the end-loads and end-displacements, have been derived from the implicit expressions (5), (6), and (10), respectively. Since these implicit expressions were shown to be consistent with each other via PVW in Section 2, the resulting explicit expressions should also be consistent with regards to PVW.

Thus, a variation of the strain energy, given by Eq.(26), keeping the external loads constant, in response to virtual displacements δu_{x1}, δu_{y1}, and $\delta\theta_{z1}$ that satisfy the geometric constraint condition (25), can be equated to the virtual work done by the external forces over these virtual displacements. This implies:

$$\begin{aligned} &\delta v = \boldsymbol{f}_{x1}\delta u_{x1} + \boldsymbol{f}_{y1}\delta u_{y1} + \boldsymbol{m}_{z1}\delta\theta_{z1} \\ &\Rightarrow \sum_{n=0}^{\infty} \boldsymbol{f}_{x1}^{\,n} \left\{ \delta u_{y1} \quad \delta\theta_{z1} \right\} \begin{bmatrix} v_{11}^{(n)} & v_{12}^{(n)} \\ v_{12}^{(n)} & v_{22}^{(n)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + k_{33} u_{x1}^{(e)} \delta u_{x1}^{(e)} \\ &= 2\sum_{n=0}^{\infty} \boldsymbol{f}_{x1}^{\,n+1} \left\{ \delta u_{y1} \quad \delta\theta_{z1} \right\} \begin{bmatrix} g_{11}^{(n)} & g_{12}^{(n)} \\ g_{12}^{(n)} & g_{22}^{(n)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ &\quad + \boldsymbol{f}_{x1}\delta u_{x1}^{(e)} + \boldsymbol{f}_{y1}\delta u_{y1} + \boldsymbol{m}_{z1}\delta\theta_{z1} \end{aligned} \tag{27}$$

Since the virtual displacements δu_{x1}, δu_{y1}, and $\delta\theta_{z1}$ are arbitrary, their respective coefficients may be set to zero. This leads to the follow end load-displacement relations:

$$\boldsymbol{f}_{x1} = k_{33} u_{x1}^{(e)} \tag{28}$$

$$\begin{aligned} \begin{Bmatrix} \boldsymbol{f}_{y1} \\ \boldsymbol{m}_{z1} \end{Bmatrix} &= \begin{bmatrix} v_{11}^{(0)} & v_{12}^{(0)} \\ v_{12}^{(0)} & v_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ &\quad + \left(\sum_{n=1}^{\infty} \boldsymbol{f}_{x1}^{\,n} \begin{bmatrix} v_{11}^{(n)} - 2g_{11}^{(n-1)} & v_{12}^{(n)} - 2g_{12}^{(n-1)} \\ v_{12}^{(n)} - 2g_{12}^{(n-1)} & v_{22}^{(n)} - 2g_{22}^{(n-1)} \end{bmatrix} \right) \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \end{aligned} \tag{29}$$

Eq.(28) is an expected result and provides no new information. However, a comparison between Eqs.(29) and (24) , both of which should be identical given the above-mentioned consistency in the energy formulation, reveals a fundamental relation between the stiffness, constraint, and energy coefficients:

$$\begin{aligned} k_{\beta\lambda}^{(0)} &= v_{\beta\lambda}^{(0)} \\ k_{\beta\lambda}^{(n)} &= v_{\beta\lambda}^{(n)} - 2g_{\beta\lambda}^{(n-1)} \qquad \forall\ n = 1..\infty \end{aligned} \tag{30}$$

where β and λ, both take indicial values of 1 and 2. This explains some of the similarities observed at the end of Section 3. The above relations may be readily verified for the case of a simple beam using known results (13), (16), and (18); however, it should be noted that these are valid for any general beam shape, as proven above.

A second argument, based on the conservation of energy, provides yet another fundamental relation between the beam characteristic coefficients. Since a given set of end-loads ($\boldsymbol{f}_{x1}$, $\boldsymbol{f}_{y1}$, and $\boldsymbol{m}_{z1}$) produces a unique set of end-displacements (u_{x1}, u_{y1}, and θ_{z1}), the resulting strain energy stored in the deformed beam is also unique, given by Eq.(26). This strain energy remains the same irrespective of the order in which the loading is carried out. Therefore, we consider a case where the loading is performed in two steps: **1.** End loads $\boldsymbol{f}_{y1}$ and $\boldsymbol{m}_{z1}$ are applied to produce some end-displacements $\bar{u}_{x1}$, u_{y1}, and θ_{z1}, and **2.** While holding the end-displacements u_{y1} and θ_{z1} fixed, end-load $\boldsymbol{f}_{x1}$ is applied to change the axial displacement from $\bar{u}_{x1}$ to u_{x1}.

The sum of energy added to the beam in these two steps should be equal to the final strain energy given by Eq.(26). Energy stored in Step 1 is simply obtained by setting $\boldsymbol{f}_{x1} = 0$ in Eq.(26)

$$v_1 = \frac{1}{2}\left\{u_{y1} \quad \theta_{z1}\right\}\begin{bmatrix} v_{11}^{(0)} & v_{12}^{(0)} \\ v_{12}^{(0)} & v_{22}^{(0)} \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \tag{31}$$

Axial displacement at the end of Step 1 is simply given by setting $\boldsymbol{f}_{x1} = 0$ in Eq.(25)

$$\bar{u}_{x1} = \left\{u_{y1} \quad \theta_{z1}\right\}\begin{bmatrix} g_{11}^{(0)} & g_{12}^{(0)} \\ g_{12}^{(0)} & g_{22}^{(0)} \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \tag{32}$$

Next, assuming a conservative system, the energy added to the beam in Step 2 may simply be determined by calculating the work done on the system when force $\boldsymbol{f}_{x1}$ causes the beam end to move from $\bar{u}_{x1}$ to u_{x1} in the axial direction. An integral needs to be carried out since the relation between $\boldsymbol{f}_{x1}$ and u_{x1} is non-linear. However, since inverting Eq.(25), which provides displacement in terms of force, is not trivial, determining the work done in this fashion is difficult, if not impossible. Therefore, instead we choose to determine the complimentary work, which is readily derived from Eq.(25)

$$v_2^*\left(\boldsymbol{f}_{x1}\right) = w_2^*\left(\boldsymbol{f}_{x1}\right) = \int_0^{\boldsymbol{f}_{x1}} \left(u_{x1} - \bar{u}_{x1}\right)\cdot d\boldsymbol{f}_{x1} \tag{33}$$

This result is then used to calculate the strain energy stored in the beam during Step 2, as follows:

$$v_2\left(u_{x1}\right) = \left(u_{x1} - \bar{u}_{x1}\right)\cdot \boldsymbol{f}_{x1} - v_2^*\left(\boldsymbol{f}_{x1}\right) \tag{34}$$

Substituting Eqs. (25) and (32) first in Eq. (33), and then all these three in Eq.(34) yields:

$$\begin{aligned} v_2 &= \boldsymbol{f}_{x1}\frac{\boldsymbol{f}_{x1}}{k_{33}} + \sum_{n=1}^{\infty} \boldsymbol{f}_{x1}^{n+1}\left\{u_{y1} \quad \theta_{z1}\right\}\begin{bmatrix} g_{11}^{(n)} & g_{12}^{(n)} \\ g_{12}^{(n)} & g_{22}^{(n)} \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ &\quad - \frac{\boldsymbol{f}_{x1}^2}{2k_{33}} - \sum_{n=1}^{\infty}\frac{1}{n+1}\boldsymbol{f}_{x1}^{n+1}\left\{u_{y1} \quad \theta_{z1}\right\}\begin{bmatrix} g_{11}^{(n)} & g_{12}^{(n)} \\ g_{12}^{(n)} & g_{22}^{(n)} \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ &= \frac{\boldsymbol{f}_{x1}^2}{2k_{33}} + \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\boldsymbol{f}_{x1}^{n+1}\left\{u_{y1} \quad \theta_{z1}\right\}\begin{bmatrix} g_{11}^{(n)} & g_{12}^{(n)} \\ g_{12}^{(n)} & g_{22}^{(n)} \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix}\right) \\ &= \frac{\boldsymbol{f}_{x1}^2}{2k_{33}} + \sum_{n=1}^{\infty}\left(\frac{n-1}{n}\boldsymbol{f}_{x1}^{n}\left\{u_{y1} \quad \theta_{z1}\right\}\begin{bmatrix} g_{11}^{(n-1)} & g_{12}^{(n-1)} \\ g_{12}^{(n-1)} & g_{22}^{(n-1)} \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix}\right) \end{aligned} \tag{35}$$

Since $v = v_1 + v_2$, Eqs. (26), (31), and (35) imply that there is another fundamental relation between the energy and constraint coefficients, given by:

$$v_{\beta\lambda}^{(n)} = 2\left(\frac{n-1}{n}\right)g_{\beta\lambda}^{(n-1)} \quad \forall\ n = 1..\infty \tag{36}$$

Expressions (30) and (36) may be further manipulated to yield the following relations between stiffness and constraint coefficients, and energy and stiffness coefficients:

$$k_{\beta\lambda}^{(n)} = -\frac{2}{n}g_{\beta\lambda}^{(n-1)} \quad \forall\ n = 1..\infty \tag{37}$$

$$\begin{aligned} v_{\beta\lambda}^{(0)} &= k_{\beta\lambda}^{(0)} \\ v_{\beta\lambda}^{(n)} &= -\left(n-1\right)k_{\beta\lambda}^{(n)} \quad \forall\ n = 1..\infty \end{aligned} \tag{38}$$

Together, Eqs. (36)-(38) present the far-reaching conclusion that the stiffness, constraint, and energy expressions are all inter-related; any one can be expressed in terms of any of the other two. These relations may be readily verified for the known case of a simple beam via Eqs. (13), (16), and (18).

The above derived relations offer considerable insight into the nature of the non-linear results for variable cross-section beam flexures. Some specific observations are noted below:

1. Eq.(38) indicates that that no matter what the beam shape is, $v_{\beta\lambda}^{(1)}$ is always zero. This simply implies that while all other stiffness coefficients contribute to the strain energy, the stiffness coefficient associated with the first power of $\boldsymbol{f}_{x1}$ does not. This agrees with our physical understanding since the stiffness coefficient $k_{\beta\lambda}^{(1)}$ represents a stiffness component arising from geometry and not displacement.

2. Eq.(37) shows that $k_{\beta\lambda}^{(1)} = -2g_{\beta\lambda}^{(0)}$, irrespective of the beam shape. This indicates that the load-stiffening effect seen in the transverse direction load-displacement relation and the kinematic component seen in the axial direction geometric constraint relation, are inherently related. This should be no surprise either, because, in physical terms, both these effects arise from the consideration of the beam in a deformed configuration.

3. These results (Eqs. (36)-(38)) also highlight the fact that the transverse load-displacement expression (24), the axial geometric constraint expression (25), and the strain energy expression (26) for a generalized beam are not entirely independent. The geometric constraint expression captures all the beam characteristic coefficients, except for the elastic stiffness $k_{\beta\lambda}^{(0)}$. The strain energy, on the other hand, captures all the beam characteristic coefficients except for load-stiffening and kinematic ones. However, the transverse direction load-displacement relation is the most complete of the three – in fact, it captures all the beam characteristic coefficients. This is reasonable because as per the PVW, both the strain energy and geometric constraint relations are used in deriving the transverse load-displacement relation. Thus, the latter captures all the information contained in the former two.

This last observation leads to an important practical advantage. It implies that in the derivation of the non-linear beam mechanics, which ultimately leads to the BCM, it is no longer necessary to determine all three relations individually. In fact, solving for the constraint and energy relations individually is mathematically more tedious because of the integration steps and the quadratic terms in u_{y1} and θ_{z1} involved. Instead, one may simply derive the transverse load-displacement relation, and determine the constraint and energy relations indirectly using Eqs.(37) and (38).

5. BCM ENERGY FORMULATION FOR INITIALLY-STRAIGHT VARIABLE CROSS-SECTION BEAMS

In this section, we employ the results from the previous two sections to present an energy formulation associated with the BCM for a variable cross-section beam. One of the questions raised at the end of Section 3 was how to determine the truncation of the transverse load-displacement (24), the axial geometric constraint (25), and the strain energy (26) expressions, all of which are expressed in the form of infinite series in the axial load $\boldsymbol{f}_{x1}$. Observation **3** from the previous section helps provide an answer. Since these three expressions are all inter-related, their truncation should be such that the expressions remain consistent in terms of PVW even after truncation. Maintaining this consistency is important because ultimately we plan to use only this truncated strain energy of a generalized beam in deriving the load-displacement relations for more complex flexure mechanisms using energy methods.

It has been identified analytically as well as experimentally [5-7] that terms up to the first power in $\boldsymbol{f}_{x1}$ have to be retained both in the constraint expression to capture the kinematic and elastokinematic effects, and in the transverse load-displacement expression to capture the elastic and load-stiffening effects. Based on these requirements, a consistent BCM, comprising transverse load-displacement, axial constraint, and strain energy relations, that captures elastic stiffness, load-stiffening, kinematic, and elastokinematic effects is given by:

$$\begin{Bmatrix} \boldsymbol{f}_{y1} \\ \boldsymbol{m}_{z1} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(0)} & k_{12}^{(0)} \\ k_{12}^{(0)} & k_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \boldsymbol{f}_{x1} \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{12}^{(1)} & k_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \boldsymbol{f}_{x1}^{2} \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{12}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \quad (39)$$

$$u_{x1} = u_{x1}^{(e)} + \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} g_{11}^{(0)} & g_{12}^{(0)} \\ g_{12}^{(0)} & g_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \boldsymbol{f}_{x1} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} g_{11}^{(1)} & g_{12}^{(1)} \\ g_{12}^{(1)} & g_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ \theta_{z1} \end{Bmatrix} \quad (40)$$

$$v = \frac{1}{2} k_{33} u_{x1}^{(e)^2} + \frac{1}{2} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} v_{11}^{(0)} & v_{12}^{(0)} \\ v_{12}^{(0)} & v_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \frac{1}{2} \boldsymbol{f}_{x1}^{2} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} v_{11}^{(2)} & v_{12}^{(2)} \\ v_{12}^{(2)} & v_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \quad (41)$$

subject to the following relations between the beam characteristic coefficients:

$$k_{\beta\lambda}^{(0)} = v_{\beta\lambda}^{(0)} \;;\; g_{\beta\gamma}^{(0)} = -\frac{1}{2} k_{\beta\gamma}^{(1)} \;;\; g_{\beta\gamma}^{(1)} = v_{\beta\lambda}^{(2)} = -k_{\beta\gamma}^{(2)} \quad (42)$$

To employ the above results in an energy method such as PVW, we need to know the strain energy of any constituent flexure beams as a function of displacements only, and similarly any geometric constraints between these displacements. However, in their present forms both the strain energy expression (41) as well as the geometric constraint expression (40) exhibit the presence of the axial load $\boldsymbol{f}_{x1}$. Therefore, further modification of these two expressions is necessary. This is simply achieved by making the logical substitution $\boldsymbol{f}_{x1} = k_{33} u_{x1}^{(e)}$ in these two equations, to yield:

$$v = \frac{1}{2} k_{33} u_{x1}^{(e)^2} \left(1 + k_{33} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} v_{11}^{(2)} & v_{12}^{(2)} \\ v_{12}^{(2)} & v_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \right) + \frac{1}{2} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} v_{11}^{(0)} & v_{12}^{(0)} \\ v_{12}^{(0)} & v_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix}$$

$$u_{x1} = u_{x1}^{(e)} \left(1 + k_{33} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} g_{11}^{(1)} & g_{12}^{(1)} \\ g_{12}^{(1)} & g_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ \theta_{z1} \end{Bmatrix} \right) + \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} g_{11}^{(0)} & g_{12}^{(0)} \\ g_{12}^{(0)} & g_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix}$$

Now these expressions are in the desirable form. The independent displacement variables in this case are $u_{x1}^{(e)}$, u_{y1}, and θ_{z1}; and u_{x1} is a dependent displacement coordinate related to the former three via the second of the above two equations. Alternatively, the constraint equation may be substituted into the strain energy expression, while employing the relations (42) , to yield:

$$v = \frac{1}{2} k_{33} \frac{\left(u_{x1} + \frac{1}{2} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{12}^{(1)} & k_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \right)^2}{\left(1 - k_{33} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{12}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ \theta_{z1} \end{Bmatrix} \right)} + \frac{1}{2} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} k_{11}^{(0)} & k_{12}^{(0)} \\ k_{12}^{(0)} & k_{22}^{(0)} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \quad (43)$$

This is the final non-linear expression for strain energy that is consistent with the BCM. In this form, the strain energy may be used directly in an energy-based analysis of multi-beam flexure mechanisms, without the need for an additional constraint expression. The constraint is implicit here, and u_{x1}, u_{y1}, and θ_{z1} are the three independent displacement variables.

6. MULTI-BEAM PARALLELOGRAM FLEXURE ANALYSIS USING THE BCM ENERGY FORMULATION

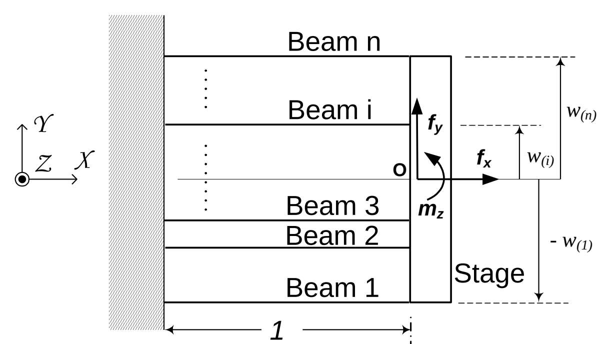

Fig.3 Multi-Beam Parallelogram Flexure

A multi-beam parallelogram flexure mechanism is shown in Fig.3. A rigid stage is connected to ground via parallel and identical beams, not necessarily uniform in thickness, numbered *1* through *n*. External loads $\boldsymbol{f}_x$, $\boldsymbol{f}_y$, and $\boldsymbol{m}_z$, normalized as per the previously described scheme, act at point O on the rigid stage. A reference line, passing through O and parallel to the beams, is used to specify the location of the i^{th} beam via the geometric parameter w_i measured along the positive Y axis. The spacing between the beams is arbitrary. The normalized displacements of point O, under the given loads, are denoted by u_x, u_y, and θ_z (not shown in the figure). It is physically obvious that the Y direction represents a DoF, while the axial direction X and transverse direction Θ_Z represent DoC given their high stiffness.

The multi-beam parallelogram flexure module allows the use of thinner beams that lead to a low DoF stiffness without compromising DoC stiffness. This ensures a larger DoF motion range along with good DoC load bearing capacity [8-10]. Consequently, one would like to study the effect of the number of beams and their spacing on stiffness and error motion behavior. This necessitates the determination of the stage displacements in terms of the three externally applied loads. A direct analysis of this system would require the creation of Free Body Diagrams for each beam, explicitly identifying its end-loads. The end load-displacement relations for each beam provide $3n$ constitutive relations, while another 3 equations are obtained from the load equilibrium of the stage in its displaced configuration. These $3(n+1)$ equations have to be solved simultaneously for the $3n$ unknown internal end-loads and the three displacements of the motion stage (u_x, u_y, and θ_z). Even though the $3n$ internal end-loads are of no direct interest, they have to be determined in this direct analysis. Obviously, the complexity associated with solving $3(n+1)$ equations grows with increasing number of beams.

Instead, an energy based approach for determining the load-displacement relations for the multi-beam parallelogram flexure turns out to be far more efficient. We first identify the geometric compatibility conditions in this case by expressing the end displacements of each beam in terms of the stage displacements. Since a physical understanding of the system as well as previous analytical results [6, 8] show that the stage angle θ_z is very small ($\sim 10^{-3}$), the small angle approximations $\cos\theta_z = 1$ and $\sin\theta_z = \theta_z$ are well-justified. Thus, the end displacements for the i^{th} beam are given by:

$$u_{x1(i)} \approx u_x - w_{(i)}\theta_z \quad ; \quad u_{y1(i)} \approx u_y \quad ; \quad \theta_{z1(i)} \approx \theta_z \tag{44}$$

Next, using Eq.(43), the strain energy for the i^{th} beam is given by:

$$v_{(i)} = \frac{1}{2}k_{33}\frac{\left(u_x - w_{(i)}\theta_z + \frac{1}{2}\begin{Bmatrix} u_y & \theta_z \end{Bmatrix}\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{12}^{(1)} & k_{22}^{(1)} \end{bmatrix}\begin{Bmatrix} u_y \\ \theta_z \end{Bmatrix}\right)^2}{\left(1 - k_{33}\begin{Bmatrix} u_y & \theta_z \end{Bmatrix}\begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{12}^{(2)} & k_{22}^{(2)} \end{bmatrix}\begin{Bmatrix} u_y \\ \theta_z \end{Bmatrix}\right)} + \frac{1}{2}\begin{Bmatrix} u_y & \theta_z \end{Bmatrix}\begin{bmatrix} k_{11}^{(0)} & k_{12}^{(0)} \\ k_{12}^{(0)} & k_{22}^{(0)} \end{bmatrix}\begin{Bmatrix} u_y \\ \theta_z \end{Bmatrix}$$

The total strain energy of the system is simply the sum of the strain energies of all the beams:

$$v = \frac{1}{2}k_{33}\frac{\sum_{i=1}^{n}\left(u_x - w_{(i)}\theta_z + \frac{1}{2}\left\{k_{11}^{(1)}u_y^2 + 2k_{12}^{(1)}u_y\theta_z + k_{22}^{(1)}\theta_z^2\right\}\right)^2}{\left(1 - k_{33}\left\{k_{11}^{(2)}u_y^2 + 2k_{12}^{(2)}u_y\theta_z + k_{22}^{(2)}\theta_z^2\right\}\right)} + \frac{1}{2}n\left\{k_{11}^{(0)}u_y^2 + 2k_{12}^{(0)}u_y\theta_z + k_{22}^{(0)}\theta_z^2\right\} \tag{46}$$

Applying the PVW, the variation of strain energy in response to virtual displacements δu_x, δu_y, and $\delta\theta_z$ may be equated to the virtual work done by external forces. In the resulting equation, the coefficients of each of these virtual displacements may be identically set to zero. This results in the following three relations, where the first one is used to simplify the subsequent two:

$$\boldsymbol{f_x} = nk_{33}\frac{\left(u_x + \frac{1}{2}\left\{k_{11}^{(1)}u_y^2 + 2k_{12}^{(1)}u_y\theta_z + k_{22}^{(1)}\theta_z^2\right\}\right) - \left(\frac{1}{n}\sum_{i=1}^{n}w_{(i)}\right)\theta_z}{\left(1 - k_{33}\left\{k_{11}^{(2)}u_y^2 + 2k_{12}^{(2)}u_y\theta_z + k_{22}^{(2)}\theta_z^2\right\}\right)} \tag{47}$$

$$\begin{aligned}\boldsymbol{f_y} = {} & n\left(k_{11}^{(0)}u_y + k_{12}^{(0)}\theta_z\right) \\ & + k_{33}\sum_{i=1}^{n}\left\{\frac{\boldsymbol{f_x}}{nk_{33}} + \frac{\left(\frac{1}{n}\sum_{i=1}^{n}w_{(i)} - w_{(i)}\right)\theta_z}{\left(1 - k_{33}\left\{k_{11}^{(2)}u_y^2 + 2k_{12}^{(2)}u_y\theta_z + k_{22}^{(2)}\theta_z^2\right\}\right)}\right\}\left(k_{11}^{(1)}u_y + k_{12}^{(1)}\theta_z\right) \\ & + k_{33}^2\sum_{i=1}^{n}\left\{\frac{\boldsymbol{f_x}}{nk_{33}} + \frac{\left(\frac{1}{n}\sum_{i=1}^{n}w_{(i)} - w_{(i)}\right)\theta_z}{\left(1 - k_{33}\left\{k_{11}^{(2)}u_y^2 + 2k_{12}^{(2)}u_y\theta_z + k_{22}^{(2)}\theta_z^2\right\}\right)}\right\}^2\left(k_{11}^{(2)}u_y + k_{12}^{(2)}\theta_z\right)\end{aligned} \tag{48}$$

$$\begin{aligned}\boldsymbol{m_z} = {} & n\left(k_{12}^{(0)}u_y + k_{22}^{(0)}\theta_z\right) \\ & + k_{33}\sum_{i=1}^{n}\left[\begin{array}{l}\left(-w_{(i)} + k_{12}^{(1)}u_y + k_{22}^{(1)}\theta_z\right)* \\ \left\{\frac{\boldsymbol{f_x}}{nk_{33}} + \frac{\left(\frac{1}{n}\sum_{i=1}^{n}w_{(i)} - w_{(i)}\right)\theta_z}{\left(1 - k_{33}\left\{k_{11}^{(2)}u_y^2 + 2k_{12}^{(2)}u_y\theta_z + k_{22}^{(2)}\theta_z^2\right\}\right)}\right\}\end{array}\right] \\ & + k_{33}^2\sum_{i=1}^{n}\left\{\frac{\boldsymbol{f_x}}{nk_{33}} + \frac{\left(\frac{1}{n}\sum_{i=1}^{n}w_{(i)} - w_{(i)}\right)\theta_z}{\left(1 - k_{33}\left\{k_{11}^{(2)}u_y^2 + 2k_{12}^{(2)}u_y\theta_z + k_{22}^{(2)}\theta_z^2\right\}\right)}\right\}^2\left(k_{12}^{(2)}u_y + k_{22}^{(2)}\theta_z\right)\end{aligned} \tag{49}$$

For a DoF motion range $u_y \sim 0.1$, Eq.(47) may be simplified by recognizing that $\theta_z << u_y$ to yield the axial direction displacement:

$$u_x = \frac{\boldsymbol{f_x}}{nk_{33}} \quad - \frac{1}{2}k_{11}^{(1)}u_y^2 \ \text{(45)} \quad - \frac{\boldsymbol{f_x}}{n}k_{11}^{(2)}u_y^2 \tag{50}$$

Clearly, the first term above is a purely elastic term arising from an axial stretching of the beams. The second term is a kinematic term, which is independent of the number of beams. The final term is an elastokinematic term. Similarly, Eq.(48) may be simplified to the following form:

$$\boldsymbol{f_y} = \left(nk_{11}^{(0)} \quad + \boldsymbol{f_x}k_{11}^{(1)} \quad + \frac{1}{n}\boldsymbol{f_x}^2k_{11}^{(2)}\right)u_y \tag{51}$$

Here, the first term may be identified to be the elastic stiffness term and the second term is a load-stiffening term, which is seen to be independent of the number of beams. The consistency of the energy formulation, described above, dictates that if the elastokinematic term in captured in Eq.(50), the third term (second power in $\boldsymbol{f}_x$) will show up in Eq.(51). At this final stage, one may choose drop this second power term because its contribution is practically negligible for typical beam shapes and load ranges of interest.

Finally, in addition to $\theta_z << u_y$, we also take into account the fact that k_{33} is several orders of magnitude larger than all the other stiffness coefficients, in simplifying Eq.(49). Additionally, the second power term in $\boldsymbol{f}_x$ are neglected and the special case of $\sum_{i=1}^{n} w_{(i)} = 0$ is assumed. The latter represents a somewhat symmetric arrangement of the beams about the reference axis, without diluting the generality of the above derivation. These assumptions and approximations lead to:

$$\theta_z = \frac{1}{\sum_{i=1}^{n} w_{(i)}^2}\left[\boldsymbol{m}_z - \left(nk_{12}^{(0)} + \boldsymbol{f}_x k_{12}^{(1)}\right)u_y\right]\left(\frac{1}{k_{33}} - k_{11}^{(2)}u_y^2\right) \tag{52}$$

The accuracy of the above closed-form parametric analytical are corroborated via non-linear FEA carried out in ANSYS. A 7-beam parallelogram flexure is selected for this FEA study, with the beam locations w_i arbitrarily chosen with respect to a reference X axis passing through the center of the stage. The beams considered are all initially straight and uniform in thickness. Each beam is *5mm* in thickness, *50mm* in height, and *250mm* in the length; the latter serves to normalize all other displacements and length dimensions. The normalized values of the w_i's selected are: *– 0.6*, *– 0.45*, *– 0.25*, *– 0.1*, *0.2*, *0.35* and *0.6*. BEAM4 elements are used for meshing, with the consistent matrix and large displacement (NLGEOM) options turned on to capture all non-linearities in the problem. A Young's modulus of *210,000 N/mm²* and Poisson's ratio of *0.3* are used assuming the material to be Steel.

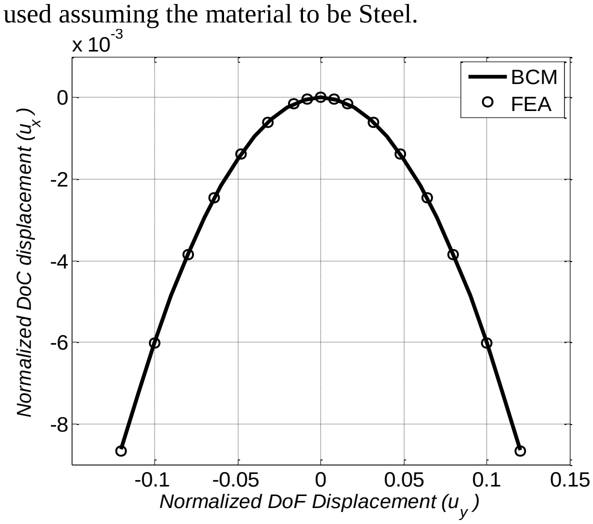

Fig.4 Parasitic axial displacement u_x (DoC) vs. transverse displacement u_y (DoF)

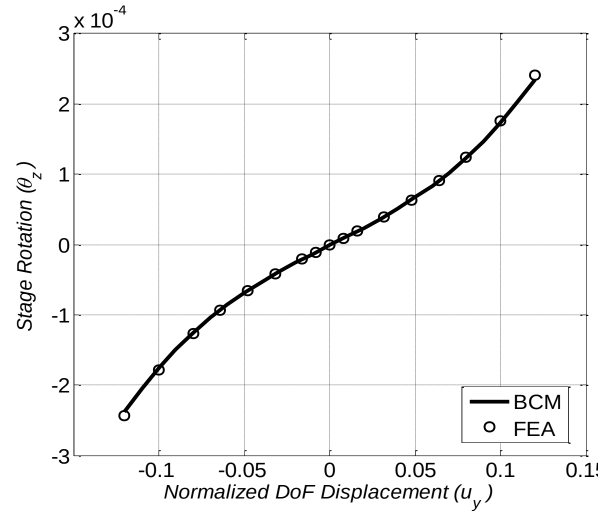

Fig.5 Parasitic stage rotation θ_z (DoC) vs. transverse displacement u_y (DoF)

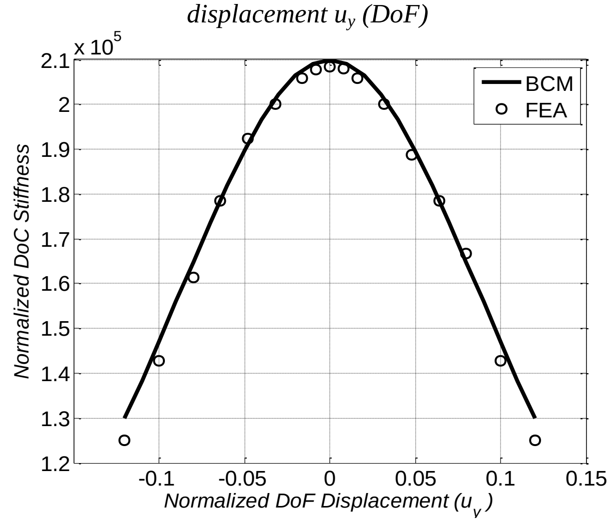

Fig.6 Axial stiffness (DoC) vs. transverse displacement u_y (DoF)

These FEA results for the 7-beam parallelogram are in agreement with the BCM predictions (Eqs.(50)-(52)), within *5%* error. This example shows that once a consistent BCM energy formulation has been derived, the use of energy methods considerably reduces the mathematical complexity in the analysis of increasingly sophisticated flexure mechanisms. The above procedure is relatively independent of the number of beams chosen or the shapes of the individual beams, as long as the strain energy associated with each beam is accounted for correctly.

7. BCM ENERGY FORMULATION FOR INITIALLY SLANTED AND CURVED BEAMS

We next consider a uniform thickness beam with an arbitrary initial angle α and an arbitrary but constant curvature κ. Fig. 7 shows such a beam with generalized end-loads and end-displacements along the X-Y-Z co-ordinate frame. All physical quantities are normalized as per the scheme described previously.

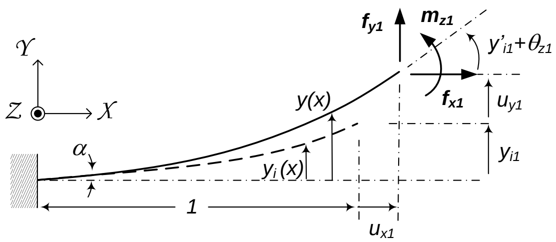

Fig. 7 Initially Slanted and Curved Beam

The initial (unloaded and undeformed) beam configuration is denoted by $y_i(x)$, final (loaded and deformed) beam configuration is given by $y(x)$, and the beam deformation in the Y direction is given by $u_y(x)$, where

$$y_i(x) = \alpha x + \frac{\kappa}{2}x^2 \text{ , and } y(x) = y_i(x) + u_y(x) \tag{53}$$

Along the previous lines, the beam governing equation may be shown to be:

$$\begin{aligned} & y''(x) - \kappa = \boldsymbol{m_{z1}} + \boldsymbol{f_{y1}}\left(1 + u_{x1} - x\right) - \boldsymbol{f_{x1}}\left(y_1 - y\left(x\right)\right) \\ & \Rightarrow y^{iv}(x) = \boldsymbol{f_{x1}} y''(x) \end{aligned} \tag{54}$$

It is to be noted that, for the curvature linearization assumption to be valid in the above equation, the initial slope α and the normalized curvature κ have to be of the order of 0.1 or less. This equation, along with boundary conditions $u_y(0)=0,\ u_y'(0)=0,\ u_y(1)=u_{y1},\ u_y'(1)=\theta_{z1}$ may be solved in closed form to determine $u_y(x)$. This solution is substituted in Eq.(10) to derive the following strain energy expression:

$$\begin{aligned} v = & \frac{1}{2}\begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} v_{11}(\boldsymbol{f_{x1}}) & v_{12}(\boldsymbol{f_{x1}}) \\ v_{21}(\boldsymbol{f_{x1}}) & v_{22}(\boldsymbol{f_{x1}}) \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + v_{44}\left(\theta_{z1} + \frac{\kappa}{2}\right)\frac{\kappa}{2} + \frac{1}{2}k_{33}\left(u_{x1}^{(e)}\right)^2 \end{aligned} \tag{55}$$

where k_{33}, v_{11}, v_{12}, and v_{22} are the same as in Eq.(17) , and

$$v_{44} = \frac{-\left(\cosh \boldsymbol{r} - 1\right)}{2\left(\boldsymbol{r}\sinh \boldsymbol{r} - 2\cosh \boldsymbol{r} + 2\right)^2} * Q_1(R) * Q_2(R)$$

$$\text{where } Q_1(R) \triangleq \left\{\boldsymbol{r}^4\left(1 + \cosh \boldsymbol{r}\right) + \boldsymbol{r}^3 \sinh \boldsymbol{r}\left(\cosh \boldsymbol{r} - 3\right)\right\}$$

$$Q_2(R) \triangleq \left\{4\boldsymbol{r}^2\left(2\cosh \boldsymbol{r} + 1\right) + 16\left(\cosh \boldsymbol{r} - 1\right) - 20\boldsymbol{r}\sinh \boldsymbol{r}\right\}$$

The above transcendental functions may be expanded to an infinite series in $\boldsymbol{f_{x1}}$ ($\triangleq \boldsymbol{r^2}$), and third power and higher terms may be truncated to yield the following compact form:

$$\begin{aligned} v = & \frac{1}{2}k_{33}\left(u_{x1}^{(e)}\right)^2 + \frac{1}{2}\begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + \frac{1}{2}\boldsymbol{f_{x1}}^2 \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} \frac{1}{700} & -\frac{1}{1400} \\ -\frac{1}{1400} & \frac{11}{6300} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + \boldsymbol{f_{x1}}^2 \frac{\kappa\theta_{z1}}{720} + \boldsymbol{f_{x1}}^2 \frac{\kappa^2}{1440} \end{aligned} \tag{56}$$

The last two terms in this strain energy expression for an initially slanted and curved beam are new as compared to Eq.(18) for an initially straight beam. Separately, the geometric constraint expression may be derived from

$$\int_0^{1+u_{x1}^{(e)}} \left\{1 + \frac{1}{2}\left(y_i'(x)\right)^2\right\}dx = \int_0^{1+u_{x1}} \left\{1 + \frac{1}{2}\left(u_y'(x) + y_i'(x)\right)^2\right\}dx \tag{57}$$

to yield a closed-form expression for the axial end-displacement u_{x1}. The resulting expression may be expanded and truncated to retain up to second power terms in $\boldsymbol{f_{x1}}$, as follows:

$$\begin{aligned} u_{x1} = & u_{x1}^{(e)} + \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{1}{20} \\ \frac{1}{20} & -\frac{1}{15} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & + \boldsymbol{f_{x1}} \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} \frac{1}{700} & -\frac{1}{1400} \\ -\frac{1}{1400} & \frac{11}{6300} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} \\ & - \left(\alpha + \frac{\kappa}{2}\right)u_{y1} - \frac{\kappa}{12}\theta_{z1} + \boldsymbol{f_{x1}}\frac{\kappa}{360}\theta_{z1} + \boldsymbol{f_{x1}}\frac{\kappa^2}{720} \end{aligned} \tag{58}$$

One may notice similarities in the energy and constraints coefficients of Eqs. (56) and (58), respectively. The reason for this can be traced back to the same arguments as were provided for an initially straight beam in Section 4. Furthermore, setting $\alpha = \kappa = 0$ reduces the above equations to those for an initially straight beam, as expected. It is interesting to note the absence of initial beam slant angle α in the strain energy expression (56) . This can be justified based on the constraint expression (58), where α is present only in the kinematic terms and not in any elastokinematic terms. Since the kinematic terms arise purely from geometry and not elastic deformation, α does not show up in the strain energy expression. The last two terms of the constraint expression, which are dependent on curvature κ, represent elastokinematic deformation and can be seen to correspond with the last two terms in the strain energy expression.

As in the case of an initially straight beam, both the constraint and strain energy expressions above have an explicit dependence on the axial load $\boldsymbol{f_{x1}}$. Employing the same arguments as presented at the end of Section 5, $\boldsymbol{f_{x1}}$ may be first replaced with $k_{33}u_{x1}^{(e)}$ in these two expressions, and $u_{x1}^{(e)}$ from the resulting constraint expression may be substituted in the resulting energy expression, to yield:

$$v = \frac{1}{2}k_{33}\frac{\left(u_{x1} - \begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{1}{20} \\ \frac{1}{20} & -\frac{1}{15} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \left(\alpha + \frac{\kappa}{2}\right)u_{y1} + \frac{\kappa}{12}\theta_{z1}\right)^2}{\left(1 + k_{33}\begin{Bmatrix} u_{y1} & \theta_{z1} \end{Bmatrix} \begin{bmatrix} \frac{1}{700} & -\frac{1}{1400} \\ -\frac{1}{1400} & \frac{11}{6300} \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix} + \tilde{k}_{33}\frac{\kappa\theta_{z1}}{360} + \tilde{k}_{33}\frac{\kappa^2}{720}\right)}$$

$$+ \quad \frac{1}{2}\left\{ u_{y1} \quad \theta_{z1} \right\} \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \end{Bmatrix}$$

This is the final non-linear strain energy expression for an initially slanted and curved beam that may be used in energy methods, with u_{x1}, u_{y1}, and θ_{z1} as the three independent displacement variables. It should be noted that while the above derivation was carried out for a uniform thickness beam, one may easily generalize it to any variable cross-section beam using the procedure outlined is Section 3. Upon such generalization, the numerical values of the energy and constraint coefficients above will be replaced by the generic symbols *v*'s and *g*'s, respectively.

8. CONCLUSION

In the past, the Beam Constraint Model (BCM) has been shown to be a dimensionless, generalized, closed-form, and parametric mathematical model that accurately captures the constraint characteristics of flexure mechanisms. These constraint characteristics are based on the stiffness and error motions in flexure elements and mechanisms, and are strongly dependent on structural non-linearities. However, the application of the BCM to more complex flexure mechanisms has proven to be tedious due to the involvement of internal loads, which are not directly relevant to the desired load-displacement relations.

The primary contribution of this paper is to provide a non-linear strain energy formulation of the BCM so that it may be employed in energy methods, such as the Principle of Virtual Work, in efficiently deriving the non-linear load-displacement relations for complex flexure mechanisms. Energy methods preclude the involvement of internal loads, thus greatly reducing mathematical complexity. We believe that this ability to accurately and quickly analyze complex flexure mechanisms is a critical first step towards their constraint-based synthesis and optimization.

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